Optimal decay rate of the compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$

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Abstract

The compressible Navier-Stokes-Poisson (NSP) system is considered in $\mathbb{R}^3$ in the present paper and the influences of the electric field of the internal electrostatic potential force governed by the self-consistent Poisson equation on the qualitative behaviors of solutions is analyzed. It is observed that the rotating effect of electric field affects the dispersion of fluids and reduces the time decay rate of solutions. Indeed, we show that the density of the NSP system converges to its equilibrium state at the same L^2-rate $(1+t)^{-\frac{3}{4}}$ or L^∞-rate $(1+t)^{-3/2}$ respectively as the compressible Navier-Stokes system, but the momentum of the NSP system decays at the L^2-rate $(1+t)^{-\frac{1}{4}}$ or L^∞-rate $(1+t)^{-1}$ respectively, which is slower than the L^2-rate $(1+t)^{-\frac{3}{4}}$ or L^∞-rate $(1+t)^{-3/2}$ for the compressible Navier-Stokes system [17, 21, 6]. These convergence rates are also shown to be optimal for the compressible NSP system.

Keywords: Compressible Navier-Stokes-Poisson, internal force, optimal decay rate.

1 Introduction and main results

The compressible Navier-Stokes-Poisson system takes the form of the Navier-Stokes equations coupled with the self-consistent Poisson equation and can be used to simulate, for instance in semiconductor devices, the transport of charged particles

under the electric field of electrostatic potential force [23]. In the present paper we consider the long time behavior of global strong solutions of the compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$. To begin with, we study the initial value problem (IVP) for the compressible Navier-Stokes-Poisson (NSP) system

$$\partial_t \rho + \nabla \cdot m = 0, \tag{1.1}$$

$$\partial_t m + \nabla \cdot (\tfrac{m \otimes m}{\rho}) + \nabla p(\rho) + \rho \nabla \Phi = \mu \Delta (\tfrac{m}{\rho}) + (\mu + \nu) \nabla (\nabla \cdot (\tfrac{m}{\rho})), \tag{1.2}$$

$$-\lambda^2 \Delta \Phi = \rho - \bar{\rho}, \quad \lim_{|x| \to \infty} \Phi(x,t) \to 0, \tag{1.3}$$

$$\rho(x,0) = \rho_0(x), \quad m(x,0) = m_0(x), \quad x \in \mathbb{R}^3. \tag{1.4}$$

The variables are the density $\rho > 0$, the momentum m, the velocity $u = \frac{m}{\rho}$, and the electrostatic potential Φ. Furthermore, $p = p(\rho)$ is the pressure function. The viscosity coefficients satisfy $\mu > 0$, $\frac{2}{3}\mu + \nu \geq 0$, and $\lambda > 0$ is the Debye length. $\bar{\rho} > 0$ denotes the background doping profile, and in this paper is taken as a positive constant for simplicity.

When there is no external or internal force involved, there are many results on the the problem of long time behavior of global smooth solutions to the compressible Navier-Stokes equations. For multi-dimensional Navier-Stokes equations, the H^s global existence and time-decay rate of strong solutions are obtained in whole space first by Matsumura-Nishida [20, 21] and the optimal L^p $(p \geq 2)$ decay rate is established by Ponce [24]. The long time decay rate of global solution in multi-dimensional half space or exterior domain is also investigated for the compressible Navier-Stokes equations by Kagei-Kobayashi [10, 11], Kobayashi-Shibata [16], and Kobayashi [15]. Therein, the optimal L^2 time-decay rate in three dimension is established as

$$\|(\rho - \bar{\rho}, u)(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}} \tag{1.5}$$

with $(\bar{\rho}, 0)$ the constant state, under small initial perturbation in Sobolev space. These time-decay rates in energy space reveal the dissipative properties of the solutions, but provide no information on wave propagation. For this reason, Zeng [29] analyzes the Green functions of one-dimensional isentropic Navier-Stokes equations and shows the L^1-convergence to the nonlinear Burgers' diffusive wave. To understand the wave propagation for compressible fluids in multi-dimension, Hoff-Zumbrun [8, 9] study the Green's function of an artificial viscosity system associated with the isentropic Navier-Stokes equations and derived the L^∞ time-decay rate of diffusive waves. Liu-Wang [17] investigate carefully the properties of the Green's function for isentropic Navier-Stokes system and present an interesting pointwise convergence of solution to diffusive waves with the optimal time-decay rate in odd dimension where the important phenomena of the weaker Huygens' principle is shown due to the stronger dispersion effects in multi-dimensional odd space. This is generalized later to the full system later in [19] where additional new waves are

introduced also. To conclude, the optimal L^∞ time-decay rate in three dimension is

$$\|(\rho-\bar\rho,u)(t)\|_{L^\infty(\mathbb{R}^3)}\le C(1+t)^{-\frac{3}{2}}. \tag{1.6}$$

The long time behavior for general multi-dimensional hyperbolic-parabolic systems is studied in [13] for multi-dimensional case with the H^s time-decay rate of solutions obtained, and in [18] for one-dimensional case with L^p $(p\ge 1)$ time-decay rate.

When additional (exterior or internal) potential force is taken into granted, the global existence of strong solution and convergence to steady state are investigated by Matsumura-Nishida [22]. As well-known, however, the external force does affect the long time behavior of dynamical solutions. The slower time-decay rate for isentropic compressible flow is investigated by Deckelnick [1, 2], which is improved later by Shibata-Tanaka [26, 27] and Ukai-Yang-Zhao [28]. The optimal L^p convergence rate in $\mathbb{R}^3$ is established recently by Duan-Ukai-Yang-Zhao [6] for the non-isentropic compressible flow as

$$\|(\rho-\tilde\rho,u,\theta-\theta_\infty)(t)\|_{L^p(\mathbb{R}^3)}\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})},\quad 2\le p\le 6, \tag{1.7}$$

where $(\tilde\rho,0,\theta_\infty)$ is related to the steady-state solution, under the same smallness assumptions on initial perturbation and the external force. The optimal L^p-L^q convergence rate with $1\le p<6/5$ and $2\le q\le 6$ in $\mathbb{R}^3$ was also established by Duan-Liu-Ukai-Yang [7] for isentropic compressible flow with external potential force.

It also should be noted that for the compressible Navier-Stokes-Poisson system related to the dynamics of self-gravitating polytropic gas, there are also important progress recently on the existence of local and global weak solutions (re-normalized solution), the reader can refer to for instance [3, 4, 25, 30], and references therein.

However, there are few results to our knowledge on the global classical solutions of Navier-Stokes-Poisson system especially for the analysis of large time behavior, besides the local and/or global existence of re-normalized weak solution in multi-dimension obtained by Donatelli [5] and Zhang-Tan [30] for Cauchy problem with different hypothesis on the pressure-density function. To this end, we first study the optimal time-decay rate of the global classical solutions to the Cauchy problem (1.1)–(1.4).

To be more precisely, the main purpose in this paper is to study the existence and uniqueness of global classical solutions and in particular the asymptotic behavior on the Cauchy problem of Navier-Stokes-Poisson system. It is observed that the electric field leads to the rotating phenomena in fluids motion, affects the speed of wave propagation and reduces the dispersion effect. This makes the Huygens' principle observed in [17] for compressible Navier-Stokes equations invalid here for the compressible compressible Navier-Stokes-Poisson system and causes the slower time-decay rate of the momentum (or velocity vector field) to the equilibrium state.

Indeed, we show in this paper that the time-decay rate of the density of the compressible Navier-Stokes-Poisson system converges to the constant state at the same algebraic time-decay rate in $\mathbb{R}^3$ (namely, $(1+t)^{-\frac{3}{4}}$ in L^2-norm or $(1+t)^{-3/2}$ in L^∞-norm respectively) as the compressible Navier-Stokes system, but the momentum decays at a slower time-rate in $\mathbb{R}^3$ (namely, $(1+t)^{-\frac{1}{4}}$ in L^2-norm or $(1+t)^{-1}$ in L^∞-norm respectively) than both the compressible Navier-Stokes system [21, 17] and the compressible Navier-Stokes system with external force [28, 6]. These convergence rates are also shown to be optimal for the compressible NSP system. To this end, we consider the linearized NSP system for density and momentum near an equilibrium state and investigate the spectral of the linear semigroup in terms of the decomposition of wave modes at lower frequency and higher frequency respectively. This makes it possible to analyze the influences of the rotating effect of electric field (caused by the internal electrostatic potential force of the self-consistent Poisson equation) on the qualitative behaviors of the global strong solutions, and finally to obtain the optimal L^p $(p \in [2,\infty])$ time decay rate of density and momentum to the original IVP problem (1.1)–(1.4) in terms of Duhamel's principle.

First, we have the following theorem on the global existence and large time behavior of classical solution to the IVP problem (1.1)–(1.4).

Theorem 1.1 *Let $p'(\rho) > 0$ for $\rho > 0$. Assume that $(\rho_0-\bar{\rho}, m_0) \in H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$, $l \geq 4$, with $\delta =: \|(\rho_0-\bar{\rho}, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}$ small. Then, there is a unique global classical solution (ρ, m, Φ) of the IVP (1.1)–(1.4) satisfying*

$$\rho - \bar{\rho} \in C^0(\mathbb{R}_+, H^l(\mathbb{R}^3)) \cap C^1(\mathbb{R}_+, H^{l-1}(\mathbb{R}^3)), \tag{1.8}$$

$$m \in C^0(\mathbb{R}_+, H^l(\mathbb{R}^3)) \cap C^1(\mathbb{R}_+, H^{l-2}(\mathbb{R}^3)), \tag{1.9}$$

$$\Phi \in C^0(\mathbb{R}_+, L^6(\mathbb{R}^3)), \quad \nabla\Phi \in C^0(\mathbb{R}_+, H^{l+1}(\mathbb{R}^3)), \tag{1.10}$$

and

$$\|\partial_x^k(\rho-\bar{\rho})(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|(\rho_0-\bar{\rho}, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}, \tag{1.11}$$

$$\|\partial_x^k m(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|(\rho_0-\bar{\rho}, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}, \tag{1.12}$$

$$\|\partial_x^k \nabla\Phi(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|(\rho_0-\bar{\rho}, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}, \tag{1.13}$$

for $k = 0, 1$, where $C > 0$ is a positive constant independent of time.

It should be noted that the time decay rates above are optimal. Indeed, we shall establish the lower bound time decay rate for the global solution.

Theorem 1.2 *Let $p'(\rho) > 0$ for $\rho > 0$. Assume $(\rho_0-\bar{\rho}, m_0) \in H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$, $l \geq 4$, with $\delta := \|(\rho_0-\bar{\rho}, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}$ small enough. Denote $n_0 =: \rho_0 - \bar{\rho}$ and assume that the Fourier transform $\widehat{n}_0 = \mathcal{F}(n_0)$ satisfies $|\widehat{n}_0(\xi)| > c_0 > 0$ for*

$0 \le |\xi| \ll 1$ *with* $c_0 > 0$ *a constant. Then, the global solution* (ρ, m, Φ) *given by Theorem 1.1 satisfies for* $t \ge t_0$ *with* $t_0 > 0$ *a sufficiently large time that*

$$c_1(1+t)^{-\frac{3}{4}} \le \|(\rho-\bar\rho)(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{3}{4}}, \tag{1.14}$$

$$c_1(1+t)^{-\frac{1}{4}} \le \|m(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{1}{4}}, \tag{1.15}$$

$$c_1(1+t)^{-\frac{1}{4}} \le \|\nabla\Phi(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{1}{4}}, \tag{1.16}$$

where $c_1, C > 0$ *are positive constants independent of time.*

With additional regularity given for the initial data, we can also prove the optimal L^p time decay rate for the global classical solution.

Theorem 1.3 *Let* $p'(\rho) > 0$ *for* $\rho > 0$*. Assume* $(\rho_0 - \bar\rho, m_0) \in H^l(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $l \ge 5$, *with* $\delta =: \|(\rho_0-\bar\rho, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}$ *small enough. Then, there is a unique global classical solution* (ρ, m, Φ) *of the IVP (1.1)–(1.4) satisfying*

$$\|(\rho-\bar\rho)(t)\|_{L^p(\mathbb{R}^3)} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}\|(\rho_0-\bar\rho, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}, \tag{1.17}$$

$$\|m(t)\|_{L^p(\mathbb{R}^3)} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}}\|(\rho_0-\bar\rho, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}, \tag{1.18}$$

$$\|\nabla\Phi(t)\|_{L^p(\mathbb{R}^3)} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}}\|(\rho_0-\bar\rho, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}, \tag{1.19}$$

for $p \in [2, \infty]$.

Remark 1.1 *It should be noted that given the same kind of initial data, the strong solution of the compressible Navier-Stokes equations exists globally in time and the momentum decays at the algebraic rate* $(1+t)^{-\frac{3}{4}}$ *in* L^2*-norm or* $(1+t)^{-\frac{3}{2}}$ *in* L^∞*-norm respectively [20, 8, 9, 17, 19], while the time-decay rate* (1.11)–(1.13) *in Theorem 1.1 and* (1.17)–(1.19) *in Theorem 1.3 implies that the momentum of the compressible Navier-Stokes-Poisson system decays at the slower time-decay rate* $(1+t)^{-\frac{1}{4}}$ *in* L^2*-norm or* $(1+t)^{-1}$ *in* L^∞*-norm. This is caused by the coupling of the electric field and velocity vector field through the Poisson equation, which also destroys the usual acoustic wave propagation for classical compressible viscous flow. As one can see in Theorem 1.2 that the time decay rate* $(1+t)^{-\frac{1}{4}}$ *of the momentum in* L^2*-norm is optimal.*

Remark 1.2 *It is also natural to show that the* L^∞*-time decay rate established in Theorem 1.3 is optimal. To this end, it is possible to show that the difference* $(\rho-\bar\rho, m, \nabla\Phi) - e^{tB}U_0$ *decays more faster than the corresponding rates in (1.11)–(1.19). In general, however, it can be carried by the analysis on the fundamental solutions (Green's function) of the linearized Navier-Stokes-Poisson system in the framework of Liu-Wang [17] under modification.*

Remark 1.3 *We also note that, the similar properties also hold for the non-isentropic NSP system, which is left in the incoming paper.*

Notations: In the following part of the paper, $C > 0$ and $c_i > 0$ with $i \geq 1$ an integer denote the generic positive constant independent of time.

The paper is arranged as follows. In section 2, we apply the spectral analysis to the semigroup for the linearized NSP system. We establish the L^2 time decay rate of the global solutions for both linearized and nonlinear NSP system in section 3. In section 4, we show the L^p time decay rate.

2 Spectral analysis of semigroup

Let us consider the IVP problem for the linearized Navier-Stokes-Poisson system

$$\partial_t n + \nabla \cdot m = 0, \tag{2.1}$$

$$\partial_t m + c^2 \nabla n + \lambda^{-2} \nabla (-\Delta)^{-1} n - \mu \Delta m - (\mu + \nu) \nabla (\nabla \cdot m) = 0, \tag{2.2}$$

$$\Phi = \lambda^{-2} (-\Delta)^{-1} n, \qquad \lim_{|x| \to \infty} \Phi(x,t) \to 0, \tag{2.3}$$

$$n(x,0) = n_0(x) =: \rho_0(x), \quad m(x,0) = m_0(x), \quad x \in \mathbb{R}^3. \tag{2.4}$$

In terms of the semigroup theory for evolutional equation, the solutions $(\bar{n}, \bar{m})$ of linear IVP problem (2.1)–(2.4) can be expressed for $\bar{U} = (\bar{n}, \bar{m})^t$ as

$$\bar{U}_t = B\bar{U}, \quad \bar{U}(0) = U_0, \quad t \geq 0, \tag{2.5}$$

which gives rise to

$$\bar{U}(t) = S(t) U_0 =: e^{tB} U_0, \quad t \geq 0. \tag{2.6}$$

What left is to analyze the differential operator B in terms of its Fourier expression A and show the long time properties of the semigroup $S(t)$. Applying the Fourier transform to system (2.5), we have

$$\partial_t \widehat{U} = A(\xi) \widehat{U}, \quad \widehat{U}(0) = \widehat{U}_0, \tag{2.7}$$

where $\widehat{U}(t) = \widehat{U}(\xi, t) = \mathcal{F} U(\xi, t)$, $\xi = (\xi_1, \xi_2, \xi_3)^t$ and $A(\xi)$ is defined as

$$A(\xi) = \begin{pmatrix} 0 & -i\xi^t \\ -i\xi(c^2 + \lambda^{-2}|\xi|^{-2}) & -\mu|\xi|^2 I_{3\times3} - (\mu+\nu)\xi \otimes \xi \end{pmatrix}. \tag{2.8}$$

The eigenvalues of the matrix A are computed from the determinant

$$det(A(\xi) - \lambda I) = (\lambda + (2\mu+\nu)|\xi|^2)^2 (\lambda^2 + (2\mu+\nu)|\xi|^2 \lambda + (c^2 + \lambda^{-2}|\xi|^{-2})|\xi|^2) = 0, \tag{2.9}$$

which implies

$$\lambda_0 = -\mu|\xi|^2, \quad (double) \tag{2.10}$$

$$\lambda_+ = -(\mu + \tfrac{1}{2}\nu)|\xi|^2 + \tfrac{1}{2} i \sqrt{4(c^2|\xi|^2 + \lambda^{-2}) - (2\mu+\nu)^2|\xi|^4}, \tag{2.11}$$

$$\lambda_- = -\,(\mu+\tfrac{1}{2}\nu)|\xi|^2 - \tfrac{1}{2}i\sqrt{4(c^2|\xi|^2+\lambda^{-2})-(2\mu+\nu)^2|\xi|^4}. \tag{2.12}$$

The semigroup e^{tA} is expressed as

$$e^{tA} = e^{\lambda_+ t}P_+ + e^{\lambda_- t}P_- + e^{\lambda_0 t}P_0 \tag{2.13}$$

where the project operators $P_0, P_\pm$ can be computed as

$$P_i = \textstyle\prod_{j\neq i} \frac{A(\xi)-\lambda_j I}{\lambda_i-\lambda_j}. \tag{2.14}$$

By a direct computation, we can verify the exact expression the Fourier transform $\widehat{G}(\xi,t)$ of Green's function $G(x,t) = e^{tB}$ as

$$\widehat{G}(\xi,t) =: e^{tA} = e^{\lambda_+ t}P_+ + e^{\lambda_- t}P_- + e^{\lambda_0 t}P_0 \tag{2.15}$$

$$= \begin{pmatrix} \frac{\lambda_+ e^{\lambda_- t}-\lambda_- e^{\lambda_+ t}}{\lambda_+-\lambda_-} & -\frac{i\xi^t(e^{\lambda_+ t}-e^{\lambda_- t})}{\lambda_+-\lambda_-} \\ -\frac{i\xi c^2(e^{\lambda_+ t}-e^{\lambda_- t})}{\lambda_+-\lambda_-} & e^{-\lambda_0 t}(I-\frac{\xi\otimes\xi}{|\xi|^2}) + \frac{\xi\otimes\xi}{|\xi|^2}\frac{(\lambda_+ e^{\lambda_+ t}-\lambda_- e^{\lambda_- t})}{\lambda_+-\lambda_-} \end{pmatrix}$$

$$+ \frac{e^{\lambda_+ t}-e^{\lambda_- t}}{\lambda_+-\lambda_-}\begin{pmatrix} 0 & 0 \\ -\frac{i\xi}{\lambda^2|\xi|^2} & 0 \end{pmatrix}. \tag{2.16}$$

To derive the long-time decay rate of solutions whatever in L^2 framework or on point-wise estimate, we need to verify the approximation of the Green's function $G(x,t)$ for both lower frequency and high frequency. In terms of the definition of the eigenvalues (2.10)–(2.12), we are able to obtain that it holds for $|\xi| \ll 1$ that

$$\frac{\lambda_+ e^{\lambda_- t}-\lambda_- e^{\lambda_+ t}}{\lambda_+-\lambda_-} \sim e^{-(\mu+\frac{1}{2}\nu)|\xi|^2 t}\left[\cos(bt) + (\mu+\tfrac{1}{2}\nu)\tfrac{\sin(bt)}{b}|\xi|^2\right], \quad |\xi| \ll 1, \tag{2.17}$$

$$\frac{\lambda_+ e^{\lambda_+ t}-\lambda_- e^{\lambda_- t}}{\lambda_+-\lambda_-} \sim e^{-(\mu+\frac{1}{2}\nu)|\xi|^2 t}\left[\cos(bt) - (\mu+\tfrac{1}{2}\nu)\tfrac{\sin(bt)}{b}|\xi|^2\right], \quad |\xi| \ll 1, \tag{2.18}$$

$$\frac{e^{\lambda_+ t}-e^{\lambda_- t}}{\lambda_+-\lambda_-} \sim \frac{\sin(tb)}{b}e^{-(\mu+\frac{1}{2}\nu)t|\xi|^2}, \quad |\xi| \ll 1, \tag{2.19}$$

where

$$b = \tfrac{1}{2}\sqrt{4(c^2|\xi|^2+\lambda^{-2})-(2\mu+\nu)^2|\xi|^4} \sim (\lambda^{-1}+\tfrac{\lambda c^2}{2}|\xi|^2)+\mathcal{O}(|\xi|^4), \quad |z| \ll 1 \tag{2.20}$$

Remark 2.4 *For the compressible Navier-Stokes equations, we have $b = c|\xi| + \mathcal{O}(|\xi|^2)$ for $|\xi| \ll 1$. Thus, the dominant behavior of wave propagation of Navier-Stokes-Poisson equations at lower frequency is different from the compressible Navier-Stokes. The so-called Huygens' principle is invalid here.*

To enclose the estimates, we also need to deal with the high frequency $|\xi| \gg 1$. In terms of the definition of the eigenvalues (2.10)–(2.12), we are able to analyze the eigenvalues for $|\xi| \gg 1$. Indeed, we have the leading orders of the eigenvalues for $|\xi| \gg 1$ as

$$\lambda_0 \sim -\,\mu|\xi|^2, \tag{2.21}$$

$$\lambda_+ \sim -(2\mu+\nu)|\xi|^2 + \frac{c^2}{2\mu+\nu} + \mathcal{O}(|\xi|^{-1}), \tag{2.22}$$

$$\lambda_- \sim -\frac{c^2}{2\mu+\nu} + \mathcal{O}(|\xi|^{-1}). \tag{2.23}$$

This approximation gives the leading order terms of the elements of Green's function as follows

$$\begin{aligned} \tfrac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} &= \tfrac{1}{2} e^{\lambda_+ t}\left[1 + e^{(\lambda_- - \lambda_+)t}\right] + \tfrac{a}{2b} e^{\lambda_+ t}\left[1 - e^{(\lambda_- - \lambda_+)t}\right] \\ &\sim \mathcal{O}(1) e^{-R_0 t},\ R_0 > 0, \quad |\xi| \geq \eta, \end{aligned} \tag{2.24}$$

$$\tfrac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \tfrac{\lambda_+}{2b} e^{\lambda_+ t}\left[1 - e^{(\lambda_- - \lambda_+)t}\right] \sim \mathcal{O}(1) e^{-R_0 t},\ R_0 > 0, \quad |\xi| \geq \eta, \tag{2.25}$$

$$\tfrac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \tfrac{1}{2b} e^{\lambda_+ t}\left[1 - e^{(\lambda_- - \lambda_+)t}\right] \sim \mathcal{O}(1) e^{-R_0 t},\ R_0 > 0, \quad |\xi| \geq \eta, \tag{2.26}$$

where R_0 and η are some positive constants.

Remark 2.5 *The dominant behavior of eigenvalues at higher frequency for the compressible Navier-Stokes-Poisson system is the same as the compressible Navier-Stokes.*

3 L^2-time decay rate

3.1 L^2 decay rate for linear semigroup

With the help of the formula (2.16) for Green's function in Fourier space and the asymptotical analysis on its elements, we are able to establish the L^2 time decay rate. Indeed, we have the L^2-time decay rate of the global strong solution to the IVP problem for the linearized Navier-Stokes-Poisson system (2.1)–(2.4) as follows.

Proposition 3.1 (L^2-theory) *Let* $U_0 = (n_0, m_0) \in H^l(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $l \geq 4$, *and denote* $(\bar{n}(t), \bar{m}(t)) =: \bar{U}(t)$. *Then,* $(\bar{n}, \bar{m}, \bar{E})$ *with* $\bar{E} = \nabla(-\Delta)^{-1}\bar{n}$ *solves the IVP* (2.1)–(2.4) *and satisfies for* $0 \leq k \leq l$ *that*

$$\left.\begin{aligned} \|\partial_x^k \bar{n}(t)\|_{L^2(\mathbb{R}^3)} &\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|U_0\|_{L^1(\mathbb{R}^3)} + \|\partial_x^k U_0\|_{L^2(\mathbb{R}^3)}), \\ \|\partial_x^k (\bar{E}, \bar{m})(t)\|_{L^2(\mathbb{R}^3)} &\leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}}(\|U_0\|_{L^1(\mathbb{R}^3)} + \|\partial_x^k U_0\|_{L^2(\mathbb{R}^3)}). \end{aligned}\right\} \tag{3.1}$$

Proof: We are going to verify that in L^2 norm, it holds

$$\|U(t)\|^2_{L^2(\mathbb{R}^3_x)} = \|\widehat{G}(\cdot,t)\widehat{U}_0\|^2_{L^2(\mathbb{R}^3_\xi)} = \|\widehat{n}(t)\|^2_{L^2(\mathbb{R}^3)} + \|\widehat{m}(t)\|^2_{L^2(\mathbb{R}^3)}. \tag{3.2}$$

and

$$\begin{aligned} \|\partial_x^k U(t)\|^2_{L^2(\mathbb{R}^3_x)} &= \||\cdot|^k \widehat{U}(\cdot,t)\|^2_{L^2(\mathbb{R}^3_\xi)} = \||\cdot|^k \widehat{G}(\cdot,t)\widehat{U}_0\|^2_{L^2(\mathbb{R}^3_\xi)} \\ &= \||\cdot|^k \widehat{n}(\xi,t)\|^2_{L^2(\mathbb{R}^3)} + \||\cdot|^k \widehat{m}(\xi,t)\|^2_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{3.3}$$

An easy computation together with the formula (2.16) of the Green's function $\widehat{G}(\xi,t)$ gives

$$
\begin{aligned}
\widehat{n}(\xi,t) =&\tfrac{\widehat{n}_0(\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t})}{\lambda_+-\lambda_-} - \tfrac{i\xi\cdot\widehat{m}_0(e^{\lambda_+t}-e^{\lambda_-t})}{\lambda_+-\lambda_-}\\
&\sim \begin{cases} e^{-(\mu+\frac12\nu)|\xi|^2t}\Big(\cos(bt)+(\mu+\frac12\nu)\frac{\sin(bt)}{b}|\xi|^2\Big)\widehat{n}_0 \\ \quad -i\xi\cdot\widehat{m}_0\frac{\sin(bt)}{b}e^{-(\mu+\frac12\nu)|\xi|^2t}, \quad |\xi|\ll 1,\\ \mathcal{O}(1)e^{-R_0t}(|\widehat{n}_0|+|\widehat{m}_0|), \quad |\xi|\gg 1,\end{cases}\\
&\sim \begin{cases} \mathcal{O}(1)e^{-(\mu+\frac12\nu)|\xi|^2t}(|\widehat{n}_0|+|\widehat{m}_0|), \quad |\xi|\ll 1,\\ \mathcal{O}(1)e^{-R_0t}(|\widehat{n}_0|+|\widehat{m}_0|), \quad |\xi|\gg 1,\end{cases}
\end{aligned}
\tag{3.4}
$$

with $R_0>0$ a constant here and below, and

$$
\begin{aligned}
\widehat{m}(\xi,t) =& -\tfrac{i\xi\widehat{n}_0c^2(e^{\lambda_+t}-e^{\lambda_-t})}{\lambda_+-\lambda_-} + \Big[\tfrac{(\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t})}{\lambda_+-\lambda_-} - e^{-\lambda_0t}\Big]\tfrac{\xi(\xi\cdot\widehat{m}_0)}{|\xi|^2}\\
&+\widehat{m}_0e^{-\lambda_0t} - \tfrac{i\xi\widehat{n}_0}{\lambda^2|\xi|^2}\tfrac{(e^{\lambda_+t}-e^{\lambda_-t})}{\lambda_+-\lambda_-}\\
\sim& \begin{cases} +e^{-\mu|\xi|^2t}\Big[e^{-\frac12\nu|\xi|^2t}\Big(\cos(bt)-(\mu+\frac12\nu)\frac{\sin(bt)}{b}|\xi|^2\Big)-1\Big]\frac{\xi(\xi\cdot\widehat{m}_0)}{|\xi|^2}\\ \quad -i\xi\widehat{n}_0(c^2+\frac{1}{\lambda^2|\xi|^2})\frac{\sin(bt)}{b}e^{-(\mu+\frac12\nu)|\xi|^2t}+\widehat{m}_0e^{-\mu|\xi|^2t}, \ |\xi|\ll 1,\\ \mathcal{O}(1)e^{-R_0t}(|\widehat{n}_0|+|\widehat{m}_0|), \quad |\xi|\gg 1,\end{cases}\\
\sim& \begin{cases} \mathcal{O}(1)e^{-\mu|\xi|^2t}(|\widehat{n}_0|+|\widehat{m}_0|)+\mathcal{O}(1)\frac{|\widehat{n}_0|}{|\xi|}e^{-\mu|\xi|^2t}, \quad |\xi|\ll 1,\\ \mathcal{O}(1)e^{-R_0t}(|\widehat{n}_0|+|\widehat{m}_0|), \quad |\xi|\gg 1,\end{cases}
\end{aligned}
\tag{3.5}
$$

where and below $\eta>0$ denotes a small but fixed constant. Therefore, we have the L^2-decay rate for $(\bar{n},\bar{m})$ as

$$
\begin{aligned}
\|\widehat{n}(t)\|^2_{L^2(\mathbb{R}^3)} =& \int_{|\xi|\le\eta}|\widehat{n}(\xi,t)|^2d\xi + \int_{|\xi|\ge\eta}|\widehat{n}(\xi,t)|^2d\xi\\
\le& C\int_{|\xi|\le\eta} e^{-(2\mu+\nu)|\xi|^2t}(|\widehat{n}_0(\xi)|^2+|\widehat{m}_0(\xi)|^2)d\xi\\
&+C\int_{|\xi|\ge\eta} e^{-R_0t}(|\widehat{n}_0(\xi)|^2+|\widehat{m}_0(\xi)|^2)d\xi\\
\le& C(1+t)^{-\frac32}\|(n_0,m_0)\|^2_{L^2(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)},
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
\|\widehat{m}(t)\|^2_{L^2(\mathbb{R}^3)} =& \int_{|\xi|\le\eta}|\widehat{m}(\xi,t)|^2d\xi + \int_{|\xi|\ge\eta}|\widehat{m}(\xi,t)|^2d\xi\\
\le& C\int_{|\xi|\le\eta} e^{-2\mu|\xi|^2t}(|\widehat{n}_0(\xi)|^2(1+|\xi|^{-2})+|\widehat{m}_0(\xi)|^2)d\xi\\
&+C\int_{|\xi|\ge\eta} e^{-R_0t}(|\widehat{n}_0(\xi)|^2+|\widehat{m}_0(\xi)|^2)d\xi\\
\le& C(1+t)^{-\frac12}\|(n_0,m_0)\|^2_{L^2(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)},
\end{aligned}
\tag{3.7}
$$

and the L^2-decay rate on the derivatives of $(\bar{n}, \bar{m})$ as

$$
\begin{aligned}
\|\widehat{\partial_x^k \bar{n}}(t)\|_{L^2(\mathbb{R}^3)}^2 =& \int_{|\xi|\le\eta} |\xi|^{2k} |\widehat{\bar{n}}(\xi,t)|^2 d\xi + \int_{|\xi|\ge\eta} |\xi|^{2k} |\widehat{\bar{n}}(\xi,t)|^2 d\xi \\
\le& C\int_{|\xi|\le\eta} e^{-(2\mu+\nu)|\xi|^2 t} |\xi|^{2k} (|\widehat{n}_0(\xi)|^2 + |\widehat{m}_0(\xi)|^2) d\xi \\
&+ C\int_{|\xi|\ge\eta} e^{-R_0 t} |\xi|^{2k} (|\widehat{n}_0(\xi)|^2 + |\widehat{m}_0(\xi)|^2) d\xi \\
\le& C(1+t)^{-\frac{3}{2}-k} (\|(n_0,m_0)\|_{L^1(\mathbb{R}^3)}^2 + \|(n_0,m_0)\|_{H^k(\mathbb{R}^3)}^2),
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\|\widehat{\partial_x^k \bar{m}}(t)\|_{L^2(\mathbb{R}^3)}^2 =& \int_{|\xi|\le\eta} |\widehat{\bar{m}}(\xi,t)|^2 d\xi + \int_{|\xi|\ge\eta} |\widehat{\bar{m}}(\xi,t)|^2 d\xi \\
\le& C\int_{|\xi|\le\eta} e^{-2\mu|\xi|^2 t} |\xi|^{2k} (|\widehat{n}_0(\xi)|^2 (1+|\xi|^{-2}) + |\widehat{m}_0(\xi)|^2) d\xi \\
&+ C\int_{|\xi|\ge\eta} e^{-R_0 t} |\xi|^{2k} (|\widehat{n}_0(\xi)|^2 + |\widehat{m}_0(\xi)|^2) d\xi \\
\le& C(1+t)^{-\frac{1}{2}-k} (\|(n_0,m_0)\|_{L^1(\mathbb{R}^3)}^2 + \|(n_0,m_0)\|_{H^k(\mathbb{R}^3)}^2),
\end{aligned}
\tag{3.9}
$$

for $1 \le k \le l$. The estimates on the $\bar{E}$ is obtained via the expression (2.3) for $0 \le k \le l$ as

$$
\begin{aligned}
\|\partial_x^k \bar{E}\|_{L^2(\mathbb{R}^3)}^2 =& C\|\partial_x^k \nabla(-\Delta)^{-1}\bar{n}\|_{L^2(\mathbb{R}^3)}^2 = C\| |\cdot|^{k-1} \widehat{\bar{n}}(t)\|_{L^2(\mathbb{R}^3)}^2 \\
\le& C(1+t)^{-\frac{1}{2}-k} (\|(n_0,m_0)\|_{L^1(\mathbb{R}^3)}^2 + \|(n_0,m_0)\|_{H^k(\mathbb{R}^3)}^2).
\end{aligned}
\tag{3.10}
$$

The proof of the Proposition 3.1 is completed. □

It should be noted that the L^2-decay rates derived above are optimal. Indeed, we have

Proposition 3.2 *Let $U_0 = (n_0, m_0) \in H^l(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and assume that the Fourier transform $\widehat{n}_0 = \mathcal{F}(n_0)$ satisfies $|\widehat{n}_0(\xi)| > c_0 > 0$ for $0 \le |\xi| \ll 1$ with c_0 a constant. Then, as $t \to +\infty$, the solution $(\bar{n}, \bar{m}, \bar{E})$ of the IVP (2.1)–(2.4) given by Proposition 3.1 satisfies*

$$
c_1(1+t)^{-\frac{3}{4}} \le \|\bar{n}(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{3}{4}}, \tag{3.11}
$$

$$
c_1(1+t)^{-\frac{1}{4}} \le \|\bar{m}(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{1}{4}}, \tag{3.12}
$$

$$
c_1(1+t)^{-\frac{1}{4}} \le \|\bar{E}(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{1}{4}}, \tag{3.13}
$$

where $c_1, C > 0$ are constants which are independent of time.

Proof: We only deal with the estimate (3.12) for simplicity, the argument applies to the others. From (3.5), we have

$$
\widehat{\bar{m}}(\xi,t) = - i\xi \widehat{n}_0 (c^2 + \tfrac{1}{\lambda^2|\xi|^2}) \tfrac{\sin(bt)}{b} e^{-(\mu+\frac{1}{2}\nu)|\xi|^2 t}
$$

$$
\begin{aligned}
&+ e^{-\mu|\xi|^2 t}\left[e^{-\frac{1}{2}\nu|\xi|^2 t}\left(\cos(bt) - (\mu+\tfrac{1}{2}\nu)\tfrac{\sin(bt)}{b}|\xi|^2\right) - 1\right]\tfrac{\xi(\xi\cdot\widehat{m}_0)}{|\xi|^2} \\
&\sim \begin{cases} \dfrac{-i\xi\widehat{n}_0}{\lambda^2|\xi|^2}\dfrac{\sin(bt)}{b}e^{-(\mu+\frac{1}{2}\nu)|\xi|^2 t} \\ \qquad + \mathcal{O}(1)e^{-\mu|\xi|^2 t}(|\widehat{n}_0| + |\widehat{m}_0|) \equiv T_1 + T_2, \quad |\xi| \ll 1, \\ \mathcal{O}(1)e^{-R_0 t}(|\widehat{n}_0| + |\widehat{m}_0|) \equiv T, \quad |\xi| \gg 1, \end{cases}
\end{aligned} \tag{3.14}
$$

with $R_0 > 0$ a constant and for lower frequency

$$
b = \frac{1}{2}\sqrt{4(c^2|\xi|^2 + \lambda^{-2}) - (2\mu+\nu)^2|\xi|^4} = (\lambda^{-1} + \tfrac{\lambda c^2}{2}|\xi|^2) + \mathcal{O}_1(|\xi|^4), \quad |\xi| \ll 1.
$$

in terms of Taylor's expansion.

Due to Parseval's equality $\|\bar{m}\| = \|\widehat{\bar{m}}\|$, it is enough to estimate the decay rate of$\|\widehat{\bar{m}}(t)\|$. It is easy to verify

$$
\begin{aligned}
\|\widehat{\bar{m}}(\xi,t)\|^2 &= \int_{|\xi|\le\eta} |\widehat{\bar{m}}(\xi,t)|^2 d\xi + \int_{|\xi|\ge\eta} |\widehat{\bar{m}}(\xi,t)|^2 d\xi \\
&\ge \int_{|\xi|\le\eta} |\widehat{\bar{m}}(\xi,t)|^2 d\xi - Ce^{-2R_0 t} \\
&\ge \int_{|\xi|\le\eta} \frac{1}{2}|T_1|^2 d\xi - \int_{|\xi|\ge\eta} |T_2|^2 d\xi - Ce^{-2R_0 t} \\
&\ge \int_{|\xi|\le\eta} \frac{1}{2}|T_1|^2 d\xi - C(1+t)^{-\frac{3}{2}} - Ce^{-2R_0 t},
\end{aligned} \tag{3.15}
$$

where and below $\eta > 0$ is a small but fixed constant. By direct computation we have

$$
\int_{|\xi|\le\eta} |T_1|^2 d\xi \ge C\int_{|\xi|\le\eta} \frac{e^{-(2\mu+\nu)|\xi|^2 t}}{|\xi|^2}\sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t + \mathcal{O}_1(|\xi|^4 t))d\xi. \tag{3.16}
$$

Applying the mean value formula we have

$$
\begin{aligned}
&\sin(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t + \mathcal{O}_1(|\xi|^4 t)) \\
&= \sin(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t) + [\sin(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t + \mathcal{O}_1(|\xi|^4 t)) - \sin(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t)] \\
&= \sin(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t) + \mathcal{O}_2(|\xi|^4 t),
\end{aligned}
$$

and then

$$
\sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t + \mathcal{O}_1(|\xi|^4 t)) \ge \frac{1}{2}\sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t) - \mathcal{O}_3([|\xi|^4 t]^2).
$$

Substituting above inequality into (3.16) we have

$$
\int_{|\xi|\le\eta} |T_1|^2 d\xi \ge C\int_{|\xi|\le\eta} \frac{e^{-(2\mu+\nu)|\xi|^2 t}}{|\xi|^2}(\sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t) - \mathcal{O}_3([|\xi|^4 t]^2))d\xi
$$

$$
\begin{aligned}
&\geq C \int_{|\xi|\leq\eta} \frac{e^{-(2\mu+\nu)|\xi|^2 t}}{|\xi|^2} \sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t) d\xi \\
&\quad - C \int_{|\xi|\leq\eta} \frac{e^{-(2\mu+\nu)|\xi|^2 t}}{|\xi|^2} (|\xi|^4 t)^2 d\xi \\
&= I_1 - I_2. \qquad (3.17)
\end{aligned}
$$

A direct computation gives rise to

$$
|I_2| \leq C \int_{|\xi|\leq\eta} \frac{e^{-(2\mu+\nu)|\xi|^2 t}}{|\xi|^2} (|\xi|^4 t)^2 d\xi \leq C(1+t)^{-\frac{5}{2}}, \qquad (3.18)
$$

and for time $t \geq t_0 =: \frac{4R^2}{\eta^2}$ with $R > \frac{\sqrt{7\pi}}{c\sqrt{\lambda}}$ that

$$
\begin{aligned}
I_1 &= C \int_{|\xi|\leq\eta} \frac{e^{-(2\mu+\nu)|\xi|^2 t}}{|\xi|^2} \sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2}|\xi|^2 t) d\xi \\
&= C t^{-\frac{1}{2}} \int_{|\zeta|\leq\eta t^{\frac{1}{2}}} e^{-(2\mu+\nu)|\zeta|^2} \sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2}|\zeta|^2) d|\zeta| \\
&= C t^{-\frac{1}{2}} \left(\int_0^R + \int_R^{\eta t^{\frac{1}{2}}} \right) e^{-(2\mu+\nu)r^2} \sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2} r^2) dr \\
&\geq c_2 (1+t)^{-\frac{1}{2}} \int_0^R e^{-(2\mu+\nu)r^2} \sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2} r^2) dr \triangleq c_2(1+t)^{-\frac{1}{2}} F(t) \qquad (3.19)
\end{aligned}
$$

with $c_2 > 0$ a positive constant.

It is easy to verify that $F(t)$ is a continuous periodic function of t with the period $\lambda\pi$. It can be also shown that there is a positive constant $F_{min} > 0$ so that

$$
F(t) \geq F_{min} =: \inf_{\frac{4R^2}{\eta^2} \leq s \leq t} F(s) > 0. \qquad (3.20)
$$

Indeed, it is trivial to note that for any $t \geq t_0 =: \frac{4R^2}{\eta^2}$ there is an integer $k_0 > 0$ so that $t \in [k_0\lambda\pi, (k_0+1)\lambda\pi]$. Thus, to show (3.20), it is sufficient to deal with $F(t)$ in one time periodic domain, namely, $t \in [k_0\lambda\pi, (k_0+1)\lambda\pi]$ for some $k_0 > 0$. We deal with the case $\lambda^{-1}t \in [k_0\pi, k_0\pi + \frac{\pi}{2}]$ and $\lambda^{-1}t \in [k_0\pi + \frac{\pi}{2}, (k_0+1)\pi]$ respectively, and obtain for $R > \frac{\sqrt{7\pi}}{c\sqrt{\lambda}}$ that

$$
\begin{aligned}
F_{min} &= \inf_{t\in[\lambda k_0\pi + \frac{\pi}{2}, \lambda(k_0+1)\pi]} \int_0^R e^{-(2\mu+\nu)r^2} \sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2} r^2) dr \\
&\geq \inf_{t\in[\lambda k_0\pi + \frac{\pi}{2}, \lambda(k_0+1)\pi]} \left(\int_{\frac{\sqrt{2\pi}}{2c\sqrt{\lambda}}}^{\frac{\sqrt{3\pi}}{2c\sqrt{\lambda}}} + \int_{\frac{\sqrt{6\pi}}{2c\sqrt{\lambda}}}^{\frac{\sqrt{7\pi}}{2c\sqrt{\lambda}}} \right) e^{-(2\mu+\nu)r^2} \sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2} r^2) dr \\
&\geq \begin{cases} \displaystyle\int_{\frac{\sqrt{2\pi}}{2c\sqrt{\lambda}}}^{\frac{\sqrt{3\pi}}{2c\sqrt{\lambda}}} e^{-(2\mu+\nu)r^2} \sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2} r^2) dr, & \lambda^{-1}t \in [k_0\pi, k_0\pi + \frac{\pi}{2}], \\ \displaystyle\int_{\frac{\sqrt{6\pi}}{2c\sqrt{\lambda}}}^{\frac{\sqrt{7\pi}}{2c\sqrt{\lambda}}} e^{-(2\mu+\nu)r^2} \sin^2(\lambda^{-1}t + \frac{\lambda c^2}{2} r^2) dr, & \lambda^{-1}t \in [k_0\pi + \frac{\pi}{2}, (k_0+1)\pi], \end{cases}
\end{aligned}
$$

$$\geq \frac{(\sqrt{3\pi}-\sqrt{2\pi})}{2c\sqrt{\lambda}} e^{-\frac{7\pi(2\mu+\nu)}{4c^2\lambda}} \sin^2\frac{\pi}{8} > 0, \quad \lambda^{-1}t \in [k_0\pi + \frac{\pi}{2}, (k_0+1)\pi]. \tag{3.21}$$

Thus, it follows from (3.19), (3.20) and (3.21) that

$$I_1 \geq c_1 F_{min}(1+t)^{-\frac{1}{2}} =: c_3(1+t)^{-\frac{1}{2}}, \quad t \geq t_0, \tag{3.22}$$

with $c_3 > 0$ a constant.

Combining (3.22), (3.15), (3.17) and (3.18), we obtain the lower bound of time-decay rate for $\bar{m}$ as

$$\|\bar{m}(t)\|_{L^2(\mathbb{R}^3)} = \|\widehat{\bar{m}}(t)\|_{L^2(\mathbb{R}^3)} \geq c_1(1+t)^{-\frac{1}{4}}, \quad t \geq t_1. \tag{3.23}$$

for some positive constants $c_1 > 0$ and $t_1 > 0$.

The time-decay rate of $\bar{n}$ and $\bar{E}$ can be shown in a similar fashion. Indeed, in view of (3.4) we need only deal with the lower frequency for the dominating term $e^{-(\mu+\frac{1}{2}\nu)|\xi|^2 t}\cos(bt)$ to obtain, after a complicated but straightforward computation, the lower bound of time-decay rate for $\bar{n}$ as

$$\|\bar{n}(t)\|_{L^2(\mathbb{R}^3)} = \|\widehat{\bar{n}}(t)\|_{L^2(\mathbb{R}^3)} \geq c_1(1+t)^{-\frac{3}{4}}. \tag{3.24}$$

We can re-present the electric field $\bar{E}$ in terms of $\bar{n}$ and the Riesz potential, and deal with the dominating term $-\frac{i\xi}{|\xi|^2}e^{-(\mu+\frac{1}{2}\nu)|\xi|^2 t}\cos(bt)$ for lower frequency to obtain the time-decay rate as

$$\|\bar{E}(t)\|_{L^2(\mathbb{R}^3)} = \|\widehat{\bar{E}}(t)\|_{L^2(\mathbb{R}^3)} \geq c_1(1+t)^{-\frac{1}{4}}, \tag{3.25}$$

where we recall that $\widehat{\bar{E}} = -\frac{i\xi}{|\xi|^2}\widehat{\bar{n}}$. The proof is completed. □

3.2 L^2 decay rate for nonlinear system

3.2.1 Reformulation of original problem

Let us reformulate the nonlinear system (1.1)–(1.4) for (ρ, u) near the equilibrium state $(\bar{\rho}, 0) = (1, 0)$. Denote

$$n = \rho - \bar{\rho}, \quad m = m, \quad \Phi = \Phi. \tag{3.26}$$

Then, the IVP problem for (n, m) is

$$\partial_t n + \nabla\cdot m = 0, \tag{3.27}$$

$$\partial_t m + c^2\nabla n + \nabla\Phi - \mu\Delta m - (\mu+\nu)\nabla(\nabla\cdot m) = -f_0, \tag{3.28}$$

$$\Phi = \lambda^{-2}(-\Delta)^{-1}n, \quad \lim_{|x|\to\infty}\Phi(x,t) \to 0, \tag{3.29}$$

$$n(x,0) = n_0(x) =: \rho_0(x) - \bar{\rho}, \quad m(x,0) = m_0(x), \quad x \in \mathbb{R}^3. \tag{3.30}$$

where $c = c(\bar{\rho}) = \sqrt{p'(\bar{\rho})}$ is the sound speed, and

$$
\begin{aligned}
f_0 = f_0(n, m, \partial_x n, \partial_x m, \partial_x^2 n, \partial_x^2 m) =: \nabla\cdot F(n, m, \partial_x n, \partial_x m), \qquad (3.31)\\
F(n, m, \partial_x n, \partial_x m) = & -\lambda^{-2}\nabla(-\Delta)^{-1} n \otimes \nabla(-\Delta)^{-1} n + \frac{1}{2}\lambda^{-2}|\nabla(-\Delta)^{-1} n|^2 I_{3\times 3} \\
& + (p(\bar{\rho}+n) - p(\bar{\rho}) - c^2 n) I_{3\times 3} \\
& + \tfrac{m\otimes m}{1+n} - \mu\nabla\big(\tfrac{nm}{1+n}\big) - (\mu+\nu)\nabla\cdot\big(\tfrac{nm}{1+n}\big) I_{3\times 3}.
\end{aligned}
$$

Denote

$$U = (n, m)^t, \quad U_0 = (n_0, m_0)^t. \qquad (3.32)$$

We have the equivalent form of system (3.27)–(3.30) in vector form

$$\partial_t U = BU + \nabla\cdot H, \quad U(0) = U_0, \qquad (3.33)$$

where the differential operator B is defined as

$$B = \begin{pmatrix} 0 & -\nabla\cdot \\ -c^2\nabla - d^{-2}\nabla(-\Delta)^{-1} & -\mu\Delta - (\mu+\nu)\nabla\nabla\cdot \end{pmatrix} \qquad (3.34)$$

and the nonlinear term H is expressed by

$$H(U, \partial_x U) = (0, F(U, \partial_x U))^t. \qquad (3.35)$$

Thus, we can represent the solution in term of the semigroup

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\nabla\cdot H(U, \partial_x U) ds \qquad (3.36)$$

with the semigroup $S(t)$ defined via multiplier through Fourier transformation

$$S(t)U = e^{tB}U = \mathcal{F}^{-1} e^{tA(\xi)} \mathcal{F} U, \quad A(\xi) = \mathcal{F}(B)(\xi), \ \xi\in\mathbb{R}^3. \qquad (3.37)$$

To establish the time decay rate of the original nonlinear problem, we need to decompose the Green's function $G =: e^{tB}$ in terms of its Fourier transform $\widehat{G}(\xi)$. Indeed, by the formula (2.16), we can make the following decomposition for $(\bar{n}, \bar{m}) = G * U_0$ as

$$\widehat{\bar{n}} = \widehat{N}\cdot\widehat{U_0} = (\widehat{\mathcal{N}} + \widehat{\mathfrak{N}})\cdot\widehat{U_0}, \quad \widehat{\bar{m}} = \widehat{M}\cdot\widehat{U_0} = (\widehat{\mathcal{M}} + \widehat{\mathfrak{M}})\cdot\widehat{U_0}, \qquad (3.38)$$

where

$$
\begin{aligned}
\widehat{\mathcal{N}} &= \begin{pmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & 0 \end{pmatrix}_{1\times 4}, \quad \widehat{\mathfrak{N}} = \begin{pmatrix} 0 & -\frac{i\xi^t (e^{\lambda_+ t} - e^{\lambda_- t})}{\lambda_+ - \lambda_-} \end{pmatrix}_{1\times 4}, \qquad (3.39)\\
\widehat{\mathcal{M}} &= \begin{pmatrix} -\frac{i\xi c^2 (e^{\lambda_+ t} - e^{\lambda_- t})}{\lambda_+ - \lambda_-} & e^{-\lambda_0 t}(I - \frac{\xi\otimes\xi}{|\xi|^2}) + \frac{\xi\otimes\xi}{|\xi|^2}\frac{(\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t})}{\lambda_+ - \lambda_-} \end{pmatrix}_{3\times 4}, \qquad (3.40)\\
\widehat{\mathfrak{M}} &= \frac{\lambda^{-2}(e^{\lambda_+ t} - e^{\lambda_- t})}{\lambda_+ - \lambda_-} \begin{pmatrix} -\frac{i\xi}{|\xi|^2} & 0 \end{pmatrix}_{3\times 4}. \qquad (3.41)
\end{aligned}
$$

And we have the Fourier expression for the electric field $\bar{E}$ as

$$\widehat{\bar{E}} = -\frac{i\xi}{|\xi|^2}\widehat{n} = (-\frac{i\xi}{|\xi|^2} \otimes \widehat{N})\widehat{U}_0 = (-\frac{i\xi}{|\xi|^2} \otimes \widehat{\mathcal{N}})\widehat{U}_0 + (-\frac{i\xi}{|\xi|^2} \otimes \widehat{\mathfrak{N}})\widehat{U}_0, \tag{3.42}$$

from which we can define

$$\widehat{\bar{E}} = \widehat{L}\widehat{U}_0 = (\widehat{\mathcal{L}} + \widehat{\mathfrak{L}})\widehat{U}_0, \tag{3.43}$$

where

$$\widehat{\mathcal{L}} = (-\frac{i\xi}{|\xi|^2} \otimes \widehat{\mathcal{N}}), \quad \widehat{\mathfrak{L}} = (-\frac{i\xi}{|\xi|^2} \otimes \widehat{\mathfrak{N}}). \tag{3.44}$$

It is easy to verify that the global solution (U, E) of the IVP problem for the nonlinear Navier-Stokes-Poisson system (3.27)–(3.30) is

$$U = (n, m) = S(t)U_0 + \int_0^t S(t-\tau)Q(U)d\tau, \quad E = \nabla(-\Delta)^{-1}n,$$

with $Q(U) = \nabla \cdot H$, which can also be decomposed as

$$n = N * U_0 + \int_0^t \mathfrak{N}(t-\tau) * Q(U)(\tau)d\tau, \tag{3.45}$$

$$m = M * U_0 + \int_0^t \mathcal{M}(t-\tau) * Q(U)(\tau)d\tau, \tag{3.46}$$

$$E = L * U_0 + \int_0^t \mathfrak{L}(t-\tau) * Q(U)(\tau)d\tau, \tag{3.47}$$

By Proposition 3.1, we have the following time decay rate for linear part

$$\|\partial_x^\alpha N * u_0(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}}(\|u_0\|_{L^1} + \|\partial_x^\alpha u_0\|_{L^2}), \tag{3.48}$$

$$\|\partial_x^\alpha M * u_0(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{2}}(\|u_0\|_{L^1} + \|\partial_x^\alpha u_0\|_{L^2}), \tag{3.49}$$

$$\|\partial_x^\alpha L * u_0(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{2}}(\|u_0\|_{L^1} + \|\partial_x^\alpha u_0\|_{L^2}), \tag{3.50}$$

for $|\alpha| \geqslant 0$. Furthermore, in view of (2.17)–(2.19) and the above definition (3.38) for $\widehat{\mathfrak{N}}, \hat{\mathcal{M}}, \widehat{\mathfrak{L}}$, it is easy to verify for some constant $c_4 > 0$ that

$$|\widehat{\mathfrak{N}}(\xi)| \sim \mathcal{O}(1)|\xi|e^{-c_4|\xi|^2 t}, \quad |\hat{\mathcal{M}}(\xi)| \sim \mathcal{O}(1)e^{-c_4|\xi|^2 t}, \quad |\widehat{\mathfrak{L}}(\xi)| \sim \mathcal{O}(1)e^{-c_4|\xi|^2 t}, \quad |\xi| \ll 1. \tag{3.51}$$

Thus, applying the similar argument as in the proof of Proposition 3.1, we are able to obtain after a straightforward computation (which we omit the details) that

$$\|\partial_x^\alpha \widehat{\mathfrak{N}} * u_0(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{1}{2}-\frac{|\alpha|}{2}}(\|u_0\|_{L^q} + \|\partial_x^\alpha u_0\|_{L^2}), \ q = 1, 2, \tag{3.52}$$

$$\|\partial_x^\alpha \hat{\mathcal{M}} * u_0(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{|\alpha|}{2}}(\|u_0\|_{L^q} + \|\partial_x^\alpha u_0\|_{L^2}), \ q = 1, 2, \tag{3.53}$$

$$\|\partial_x^\alpha \widehat{\mathfrak{L}} * u_0(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{|\alpha|}{2}}(\|u_0\|_{L^q} + \|\partial_x^\alpha u_0\|_{L^2}), \ q = 1, 2. \tag{3.54}$$

3.2.2 Global existence and L^2-decay rate

We are now ready to prove Theorem 1.1 and Theorem 1.2 on the optimal time decay rate of solution to the IVP problem of the nonlinear Navier-Stokes-Poisson system (3.27)–(3.30). From (3.35) we have

$$Q(U) = \nabla \cdot H(U, \nabla U) = Q_1 + Q_2 + Q_3, \tag{3.55}$$

which implies for smooth solution (ρ, m) satisfying $\|(\rho - \bar{\rho}, m)\|_{H^4} < \infty$ that

$$Q_1 := Q_1(U) \sim \mathcal{O}(1)(\partial_x m \cdot m), \tag{3.56}$$
$$Q_2 := Q_2(U) \sim \mathcal{O}(1)\partial_x H_2, \quad H_2 = \partial_x(m \cdot n), \tag{3.57}$$
$$Q_3 := Q_3(U) \sim \mathcal{O}(1)\partial_x n \cdot n + \mathcal{O}(1)(n \cdot E). \tag{3.58}$$

Framework of short time existence: First of all, we give the local existence theory which can be established in the framework as [21, 20, 22]. Indeed, starting with the equations (1.1)–(1.4), we can make use of the theorem of contracting map to establish the local existence. The key point is that the electric field E can be expressed by (1.1), (1.3) and the Riesz potential as a nonlocal term

$$E = E_0 + \nabla(-\triangle^{-1})div \int_0^t m ds = E_0 + \nabla(-\triangle^{-1})div \int_0^t \rho u ds,$$

which together with the L^p estimates leads to

$$\| \nabla(-\triangle^{-1})div \int_0^t \rho u ds \|_{H^k} \leq C \| \int_0^t \rho u ds \|_{H^k} .$$

Then, by the standard argument of contracting map theorem as [21, 20, 22], one can obtain the short time existence of strong solution. The details are omitted.

To extend the short time strong solution to be a global in time solution, we need to establish the uniform a-priori estimates. Indeed, under some a-priori assumptions we are able to obtain the expected estimates with time decay rate for lower order terms, and finally enclose the a-priori assumptions. We have

Lemma 3.3 *Under the assumptions of Theorem 1.1, the solution* (n, m, E) *with* $E = \nabla\Phi$ *of the IVP problem* (3.27)–(3.30) *satisfies for* $l = 4$ *that*

$$\|\partial_x^k n(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad k = 0, 1, \qquad \|\partial_x^2 n(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}},$$
$$\|\partial_x^k m(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \quad k = 0, 1, \qquad \|\partial_x^2 m(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}},$$
$$\|\partial_x^k \Phi(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \quad k = 1, 2, \qquad \|\partial_x^3 \Phi(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}},$$

and for $l = 5$ *that*

$$\|\partial_x^k n(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad k = 0, 1, 2, \qquad \|\partial_x^3 n(t)\|_{L^2} \leq C(1+t)^{-1},$$
$$\|\partial_x^k m(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \quad k = 0, 1, 2, \qquad \|\partial_x^3 m(t)\|_{L^2} \leq C(1+t)^{-1},$$
$$\|\partial_x^k \Phi(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \quad k = 1, 2, 3, \qquad \|\partial_x^4 \Phi(t)\|_{L^2} \leq C(1+t)^{-1},$$

where C *is a positive constant independent of time.*

Proof: Suppose that $(n,m,E)\in(H^4)^2\times H^5$ and $(n,m,E)\in(H^5)^2\times H^6$ with $E=\nabla\Phi$ the electric field correspond respectively to the strong solutions of the compressible Navier-Stokes-Poisson system for $t\in[0,T]$ subject to initial data in different Sobolev space. Assume that the classical solution of the IVP (3.27)–(3.30) exists for $t\in[0,T]$ and denote

$$\begin{aligned}\Lambda_1(t):=\sup_{0\le s\le t,\,k=0,1}&\{\|D_x^k n(s)\|(1+s)^{\frac{3}{4}+\frac{k}{2}}+\|D_x^k m(s)\|(1+s)^{\frac{1}{4}+\frac{k}{2}}+\|D_x^2(n,m)(s)\|(1+s)^{\frac{3}{4}}\\&+\|E(s)\|(1+s)^{\frac{1}{4}}+\|(D_x^3n,D_x^4n,D_x^3m,D_x^4m)(s)\|\},\end{aligned}\tag{3.59}$$

and

$$\begin{aligned}\Lambda_2(t):=\sup_{0\le s\le t,\,k=0,1,2}&\{\|D_x^k n(s)\|(1+s)^{\frac{3}{4}+\frac{k}{2}}+\|D_x^3 n(s)\|(1+s)+\|(D_x^4n,D_x^5n,D_x^4m,D_x^5m)(s)\|\\&+\|D_x^3m(s)\|(1+s)^{\frac{3}{4}}+\|E(s)\|(1+s)^{\frac{1}{4}}+\|D_x^k m(s)\|(1+s)^{\frac{1}{4}+\frac{k}{2}}\}.\end{aligned}\tag{3.60}$$

We claim that it holds for any $t\in[0,T]$ that

$$\Lambda_1(t)\le C\delta,\quad \Lambda_2(t)\le C\delta,\tag{3.61}$$

with δ defined in Theorem 1.1. In addition, the claim (3.61) together with the smallness assumption on δ are sufficient for us to prove the Lemma 3.3 and Theorem 1.1.

Next, let us prove the claim (3.61). It is sufficient to show the first one in (3.61) since the arguments can be also applied to second one in (3.61) under minor modifications. The proof of the first one in (3.61) consists of following three steps.

Step 1: The basic energy estimates. Starting with (3.45), Proposition 3.1, (3.48), (3.52), and the a-priori assumption (3.59), we have after a complicated but straightforward computation that

$$\begin{aligned}\|(n-N*U_0)(t)\|&\le\int_0^t\|\mathfrak{N}(t-\tau)*Q(U)(\tau)\|d\tau\\&\le C\int_0^t(1+t-\tau)^{-\frac{3}{4}-\frac{1}{2}}(\|Q(U)(\tau)\|+\|Q(U)(\tau)\|_{L^1})d\tau\\&\le C\int_0^t(1+t-\tau)^{-\frac{3}{4}-\frac{1}{2}}(\Lambda_1(t))^2(1+\tau)^{-1}d\tau\\&\le C(1+t)^{-1}(\Lambda_1(t))^2,\end{aligned}\tag{3.62}$$

where we have made use of (3.59) and (3.56)–(3.58) to estimate the right hand side terms as

$$\begin{aligned}(\|Q(U)\|+\|Q(U)\|_{L^1})&\le\{\|(Q_1+Q_2+Q_3)\|\}+\{\|(Q_1+Q_2+Q_3)\|_{L^1}\}\\&\le C\{\|Dm\|\|m\|_{L^\infty}+\|D^2m\|\|n\|_{L^\infty}+\|m\|_{L^\infty}\|D^2n\|\\&\quad+\|n\|\|E\|_{L^\infty}+\|Dn\|\|n\|_{L^\infty}\}\end{aligned}$$

$$
\begin{aligned}
&+ C\{\|Dm\|\|m\| + \|D^2m\|\|n\| + \|Dm\|\|Dn\| + \|m\|\|D^2n\| \\
&\quad + \|n\|\|E\| + \|Dn\|\|n\|\} \\
&\leq C(1+t)^{-\frac{3}{2}}(\Lambda_1(t))^2 + C(1+t)^{-1}(\Lambda_1(t))^2. \qquad (3.63)
\end{aligned}
$$

In a similar fashion, we are able to estimate the high order derivatives for density as follows. Indeed, in terms of (3.52), the Hölder's inequality and Nirenberg's inequality

$$\|u\|_{L^\infty} \leq C\|Du\|^{\frac{1}{2}}\|D^2u\|^{\frac{1}{2}}$$

and the fact

$$\|D^kE\| \leq C\|D^{k-1}n\|, \quad k \geq 1 \qquad (3.64)$$

due to the Riesz potential representation, we can estimate the term Dn as

$$
\begin{aligned}
\|Dn(t)\| &\leq \|D(N * U_0)(t)\| + \int_0^t \|D(\mathfrak{N}(t-\tau) * Q(U))(\tau)\|d\tau \\
&\leq C\delta(1+t)^{-\frac{5}{4}} + \int_0^{\frac{t}{2}} \|D(\mathfrak{N}(t-\tau) * Q(U)(\tau))\|d\tau + \int_{\frac{t}{2}}^t \|D(\mathfrak{N}(t-\tau) * Q(U)(\tau))\|d\tau \\
&\leq C\delta(1+t)^{-\frac{5}{4}} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{7}{4}}(\|Q(U)(\tau)\|_{L^1} + \|DQ(U)(\tau)\|)d\tau \\
&\quad + C\int_{\frac{t}{2}}^t \|\mathfrak{N}(t-\tau) * DQ(U)(\tau)\|d\tau \\
&\leq C\delta(1+t)^{-\frac{5}{4}} + C(\Lambda_1(t))^2\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{7}{4}}(1+\tau)^{-1}d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (1+t-\tau)^{-1}\|Q(U)(\tau)\|d\tau \\
&\leq C\delta(1+t)^{-\frac{5}{4}} + C(\Lambda_1(t))^2\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{7}{4}}(1+\tau)^{-1}d\tau \\
&\quad + C(\Lambda_1(t))^2\int_{\frac{t}{2}}^t (1+t-\tau)^{-1}(1+\tau)^{-\frac{3}{2}}d\tau \\
&\leq C\delta(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}-\varepsilon}(\Lambda_1(t))^2, \qquad (3.65)
\end{aligned}
$$

where and below $\varepsilon > 0$ is a small but fixed constant. As for D^2n, it is easy to verify that the nonlinear terms in right hand side dominating the time-decay rate consists of $D(D^2(mn))$, $D^2(Dm\cdot m)$ and $D^2(D^2(mn))$, which can be estimated due to the following facts

$$
\begin{aligned}
\|D^3n\cdot m(t)\| &\leq \|D^3n(t)\|\|m(t)\|_{L^\infty} \leq C\delta(1+t)^{-\frac{3}{4}}\Lambda_1(t), \\
\|D^3m\cdot m(t)\| &\leq \|D^3m(t)\|\|m(t)\|_{L^\infty} \leq C\delta(1+t)^{-\frac{3}{4}}\Lambda_1(t), \\
\|D^4n\cdot m(t)\| &\leq \|D^4n(t)\|\|m(t)\|_{L^\infty} \leq C\delta(1+t)^{-\frac{3}{4}}\Lambda_1(t).
\end{aligned}
$$

Thus, we can obtain after a straightforward computation that

$$\|D^2n(t)\| \leq \|D^2(N * U_0)(t)\| + \int_0^t \|D^2(\mathfrak{N}(t-\tau) * Q(U))(\tau)\|d\tau$$

$$\begin{aligned}
&\le C\delta(1+t)^{-\frac{7}{4}} + \int_0^{\frac{t}{2}} \|D^2(\mathfrak{N}(t-\tau) * Q(U)(\tau))\| d\tau \\
&\quad + \int_{\frac{t}{2}}^t \|D^2(\mathfrak{N}(t-\tau) * Q(U)(\tau))\| d\tau \\
&\le C\delta(1+t)^{-\frac{7}{4}} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{9}{4}}(\|Q(U)(\tau)\|_{L^1} + \|D^2Q(U)(\tau)\|) d\tau \\
&\quad + \int_{\frac{t}{2}}^t \|(1+t-\tau)^{-\frac{3}{2}}(\|Q(U(\tau))\| + \|D^2Q(U(\tau))\|) d\tau. \\
&\le C\delta(1+t)^{-\frac{7}{4}} + C(\Lambda_1(t))^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{9}{4}}(1+\tau)^{-\frac{3}{4}} d\tau \\
&+ C\delta\Lambda_1(t) \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{2}}(1+t)^{-\frac{3}{4}} d\tau \\
&\le C\delta(1+t)^{-\frac{7}{4}} + C(\Lambda_1(t))^2(1+t)^{-\frac{7}{4}} + C\delta\Lambda_1(t)(1+t)^{-\frac{3}{4}},
\end{aligned} \tag{3.66}$$

Next, in terms of Proposition 3.1, (3.46), (3.49), (3.53), the a-priori assumption (3.59), the Hölder's and Nirenberg's inequalities, we can prove the time decay rate for m and its derivatives as follows.

$$\begin{aligned}
\|(m - M * U_0)(t)\| &\le \int_0^t \|\mathcal{M}(t-\tau) * Q(U)(\tau)\| d\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\|Q(U)(\tau)\| + \|Q(U)(\tau)\|_{L^1}) d\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\Lambda_1(t))^2(1+\tau)^{-1} d\tau \\
&\le C(1+t)^{-\frac{1}{4}-\varepsilon}(\Lambda_1(t))^2,
\end{aligned} \tag{3.67}$$

where and below $\varepsilon > 0$ is a small but fixed constant, and

$$\begin{aligned}
\|Dm(t)\| &\le \|D(M * U_0)(t)\| + \int_0^t \|D(\mathcal{M}(t-\tau) * Q(U)(\tau))\| d\tau \\
&\le C\delta(1+t)^{-\frac{3}{4}} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{1}{2}}(\|Q(U)(\tau)\|_{L^1} + \|DQ(U)(\tau)\|)(d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}-\frac{1}{2}}(\|Q(U)(\tau)\|_{L^1} + \|DQ(U)(\tau)\|) d\tau \\
&\le C\delta(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}-\varepsilon}(\Lambda_1(t))^2,
\end{aligned} \tag{3.68}$$

where we have used (3.63). As for D^2m, we have

$$\begin{aligned}
\|D^2m(t)\| &\le \|D^2(M * U_0)(t)\| + \int_0^t \|D^2(\mathcal{M}(t-\tau) * Q(U)(\tau))\| d\tau \\
&\le C\delta(1+t)^{-\frac{5}{4}} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{7}{4}}(\|Q(U)(\tau)\|_{L^1} + \|D^2Q(U)(\tau)\|) d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}-\frac{1}{2}}(\|Q(U)(\tau)\|_{L^1} + \|D^2Q(U)(\tau)\|) d\tau
\end{aligned}$$

$$\leq C\delta(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{6}{4}-\varepsilon}(\Lambda_1(t))^2 + C\delta\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{4}-\frac{1}{2}}(1+\tau)^{-\frac{3}{4}}d\tau$$
$$\leq C\delta\Lambda_1(t)(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{6}{4}-\varepsilon}(\Lambda_1(t))^2 + C\delta\Lambda_1(t)(1+t)^{-\frac{3}{4}}. \tag{3.69}$$

What left is to obtain the time-decay rate for E in terms of (3.64) and (3.47). Indeed, it is easy to get

$$\begin{aligned}\|(E - L * U_0)(t)\| &\leq \int_0^t \|\mathfrak{L}(t-\tau) * Q(U)(\tau)\|d\tau \\ &\leq C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\|Q(U)(\tau)\| + \|Q(U)(\tau)\|_{L^1})d\tau \\ &\leq C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\Lambda_1(t))^2(1+\tau)^{-1}d\tau \\ &\leq C(1+t)^{-\frac{1}{4}-\varepsilon}(\Lambda_1(t))^2. \end{aligned} \tag{3.70}$$

Step 2: The higher order energy estimates. To enclose the a-priori estimates and prove the claim (3.61), we need to derive the time decay rate of (n, m) with respect to higher order derivatives as in [21, 20, 22]. To this end, we look on the compressible Navier-Stokes-Poisson system (3.27)–(3.29) as the compressible Navier-Stokes with nonlinear inhomogeneous terms related to the electric field E. Indeed, by (1.1)–(1.4), we derive the system for $(\rho, u) = (1+n, \frac{m}{\rho})$

$$\partial_t n + \nabla\cdot u = f_1, \tag{3.71}$$
$$\partial_t u + \nabla n + E - \mu_1\Delta u - \mu_2\nabla(\nabla\cdot u) = -f_2 - f_3, \tag{3.72}$$
$$E = \nabla(-\Delta)^{-1}n, \qquad \lim_{|x|\to\infty} E \to 0, \tag{3.73}$$
$$n(x,0) = n_0(x) =: \rho_0(x) - 1, \quad u(x,0) = u_0(x) = m_0(x)/\rho_0(x), \quad x\in\mathbb{R}^3, \tag{3.74}$$

where

$$\begin{aligned} f_1 &= f_1(n, u, \partial_x n, \partial_x u) =: -n\nabla\cdot u - \nabla n\cdot u, \\ f_2 &= f_2(n, m, \partial_x n, \partial_x m, \partial_x^2 n, \partial_x^2 m) \\ &=: (u\cdot\nabla)u + (1 - \frac{p'(1+n)}{1+n})\nabla n + \mu_1(\frac{n}{1+n})\Delta u + \mu_2(\frac{n}{1+n})\nabla(\nabla\cdot u), \\ f_3 &= -n\nabla\Phi = -nE, \end{aligned} \tag{3.75}$$

and we have chosen $\bar\rho = 1, p'(1) = 1, \mu = \mu_1, (\mu+\nu) = \mu_2$ and $d = 1$, for simplicity. Notice that, in view of (3.71) and (3.72), there are only two terms different from the classical Navier-Stokes equations, that is, E and nE.

Taking $\int(3.71)\times n dx + \int(3.72)\cdot u dx$ and integrating the resulted equation by parts, and making use of the facts $E = \nabla\Phi$ and

$$\int\nabla\Phi\cdot u dx = -\int\Phi\nabla\cdot u dx = \int\Phi n_t dx + \int\Phi f_1 dx$$

$$
\begin{aligned}
&= -\int \Phi\nabla\cdot E_t dx + \int \Phi f_1 dx = \int E\cdot E_t dx + \int \Phi f_1 dx \\
&\geq \frac{1}{2}\frac{d}{dt}\int |E|^2 dx - \|\Phi\|_{L^6}\|f_1\|_{L^{\frac{6}{5}}} \geq \frac{1}{2}\frac{d}{dt}\int |E|^2 dx - C\|E\|\|f_1\|_{L^{\frac{6}{5}}}
\end{aligned}
\tag{3.76}
$$

due to the Hölder's inequality and Sobolev inequality, we can obtain after a straightforward computation that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (|n|^2 + |u|^2 + |E|^2)(t) dx + C\int |Du(t)|^2 dx \\
\leq &\int f_1 n - (f_2 + f_3)\cdot u dx + C\|E\|\|f_1\|_{L^{\frac{6}{5}}}.
\end{aligned}
\tag{3.77}
$$

Since it is easy to verify

$$
\begin{aligned}
\|f_1\|_{L^{\frac{6}{5}}} &\leq C\|(nDu + uDn)\|_{L^{\frac{6}{5}}} \leq C\|n\|_{L^3}\|Du\|_{L^2} + C\|Dn\|_{L^2}\|u\|_{L^3} \\
&\leq C(\|n\|_{L^2} + \|n\|_{L^6})\|Du\|_{L^2} + C\|Dn\|_{L^2}(\|u\|_{L^2} + \|u\|_{L^6}) \\
&\leq C(\|n\| + \|Dn\|)\|Du\| + C\|Dn\|\|u\|,
\end{aligned}
\tag{3.78}
$$

we can control the last term on the right hand side of (3.77) as

$$
\begin{aligned}
\|E\|\|f_1\|_{L^{\frac{6}{5}}} &\leq C\|E\|(\|n\|^2 + \|Dn\|^2 + \|Du\|^2) + C\|Dn\|(\|u\|^2 + \|E\|^2) \\
&\leq C\|E\|\|(n, Dn, Du)\|^2 + C\|Dn\|\|(u, E)\|^2.
\end{aligned}
\tag{3.79}
$$

Substituting (3.79) into (3.77) we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (|n|^2 + |u|^2 + |E|^2)(t) dx + C\|Du(t)\|^2 dx \\
\leq &C\|E(t)\|\|(n, Dn, Du)(t)\|^2 + C\|Dn\|\|(u, E)(t)\|^2 + \int f_1 n - (f_2 + f_3)\cdot u dx \\
\leq &C\Lambda_1(t)\|(n, Dn, Du)(t)\|^2 + C(1+t)^{-\frac{5}{4}}\Lambda_1(t)\|(u, E)(t)\|^2 \\
&+ \int f_1 n - (f_2 + f_3)\cdot u dx.
\end{aligned}
\tag{3.80}
$$

Due to the facts

$$
f_1 = -n\nabla\cdot u - \nabla n\cdot u, \quad f_2 \sim \mathcal{O}(1)(nDn + uDu + nD^2u), \quad f_3 = nE,
$$

it is easy to estimate the last term on the right hand side of (3.80) as

$$
\begin{aligned}
&\int f_1 n - (f_2 + f_3)\cdot u dx \\
\leq &C\varepsilon\|Du(t)\|^2 + C\Lambda_1(t)\|(n, Dn, Du)(t)\|^2 + C\frac{1}{\varepsilon}\|u(t)\|_{L^\infty}^2\|u(t)\|^2 \\
\leq &C(\varepsilon\|Du(t)\|^2 + \Lambda_1(t)\|(n, Dn, Du)(t)\|^2 + \frac{1}{\varepsilon}(1+t)^{-\frac{3}{2}}\Lambda_1(t)\|u(t)\|^2),
\end{aligned}
\tag{3.81}
$$

which together with (3.80) and the smallness of constant $\varepsilon > 0$ gives rise to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(|n|^2+|u|^2+|E|^2)(t)dx + C\|Du(t)\|^2dx\\ \le& C\Lambda_1(t)\|(n,Dn,Du)(t)\|^2 + C(1+t)^{-\frac{5}{4}}\Lambda_1(t)\|(u,E)(t)\|^2\\ &+C(1+t)^{-\frac{3}{2}}\Lambda_1(t)\|u(t)\|^2. \end{aligned}\tag{3.82}$$

In a similar procedure, we are able to deal with the higher order derivatives of the solution as follows

$$\begin{aligned}&\frac{d}{dt}\int(|D^kn|^2+|D^ku|^2+|D^kE|^2)(t)dx + C\|D^{k+1}u(t)\|^2dx\\ \le& C\Lambda_1(t)(\|n(t)\|^2_{H^k}+\|Du(t)\|^2_{H^k})\\ &+C(1+t)^{-\frac{5}{4}}\Lambda_1(t)\|(u,E)(t)\|^2+C(1+t)^{-\frac{3}{2}}\Lambda_1(t)\|u(t)\|^2, \end{aligned}\tag{3.83}$$

with $k = 1, 2, 3, 4$, which implies

$$\begin{aligned}&\frac{d}{dt}(\|n(t)\|^2_{H^4}+\|u(t)(t)\|^2_{H^4}+\|E(t)(t)\|^2_{H^4}) + C\|Du(t)\|^2_{H^4}\\ \le& C\Lambda_1(t)(\|n(t)\|^2_{H^4}+\|Du(t)\|^2_{H^4})\\ &+C(1+t)^{-\frac{5}{4}}\Lambda_1(t)\|(u,E)(t)\|^2+C(1+t)^{-\frac{3}{2}}\Lambda_1(t)\|u(t)\|^2. \end{aligned}\tag{3.84}$$

Taking inner product between (3.72) and ∇n, integrating the resulted equation by parts over $\mathbb{R}^3$, and making use of $\int E\cdot\nabla n dx = -\int\nabla\cdot E n dx = \int |n|^2dx$, we have after a straightforward computation that

$$\begin{aligned}&\frac{d}{dt}\int u\cdot\nabla n dx - \int u\nabla n_t dx + \int|\nabla n|^2+|n|^2dx\\ &\le (\frac{1}{2}+\Lambda_1(t))\|(n,Dn)(t)\|^2 + C(1+\Lambda_1(t))\|(Du,D^2u)(t)\|^2, \end{aligned}\tag{3.85}$$

which leads to

$$\begin{aligned}&\frac{d}{dt}\int u\cdot\nabla n dx + \int|\nabla n|^2+|n|^2dx\\ &\le C\int u\nabla n_t dx + C(1+\Lambda_1(t))\|(Du,D^2u)(t)\|^2\\ &\le C\int u\nabla(\nabla\cdot u + n\nabla\cdot u + u\nabla n)dx + C(1+\Lambda_1(t))\|(Du,D^2u)(t)\|^2\\ &\le C\Lambda_1(t)\|(n,Dn)(t)\|^2 + C(1+\Lambda_1(t))\|Du(t)\|^2. \end{aligned}\tag{3.86}$$

Similarly, taking $\int D^\alpha(3.72)\cdot\nabla D^\alpha n dx$ with $1\le|\alpha|\le 3$, we can have after a tedious and complicated calculation that

$$\begin{aligned}&\frac{d}{dt}\int D^\alpha u\cdot\nabla D^\alpha n dx + \int|\nabla D^\alpha n|^2+|D^\alpha n|^2dx\\ \le& C\Lambda_1(t)\|n(t)\|^2_{H^4} + C(1+\Lambda_1(t))\|Du(t)\|^2_{H^3}, \end{aligned}\tag{3.87}$$

which together with (3.86) and the a-priori smallness assumption of $\Lambda_1(t)$ gives rise to

$$\frac{d}{dt}\int D^\alpha u\cdot\nabla D^\alpha n dx+\|n(t)\|_{H^4}^2\le C\|Du(t)\|_{H^3}^2,\quad 0\le|\alpha|\le 3. \tag{3.88}$$

Taking the summation (3.84) + $\beta\times$ (3.88) with $\beta>0$ small enough, and with $\Lambda(t)$ rather small we have

$$\begin{aligned}&\frac{d}{dt}K(t)+C(\|n(t)\|_{H^4}^2+\|Du(t)\|_{H^4}^2)\\ \le&C(1+t)^{-\frac{5}{4}}\Lambda_1(t)\|(u,E)(t)\|^2+C(1+t)^{-\frac{3}{2}}\Lambda_1(t)\|u(t)\|^2\\ \le&C(1+t)^{-\frac{5}{4}}\Lambda_1(t)K(t),\end{aligned} \tag{3.89}$$

where

$$K(t)=(\|n(t)\|_{H^4}^2+\|u(t)(t)\|_{H^4}^2+\|E(t)(t)\|_{H^4}^2+\beta\int\sum_{|\alpha|\le 3}D^\alpha u\cdot\nabla D^\alpha n dx)$$

and

$$C(\|n(t)\|_{H^4}^2+\|u(t)\|_{H^4}^2+\|E(t)\|_{H^4}^2)\le K(t)\le C'(\|n(t)\|_{H^4}^2+\|u(t)\|_{H^4}^2+\|E(t)\|_{H^4}^2).$$

From (3.89) we have by the Gronwall's inequality that

$$K(t)\le Ce^{\int_0^t(1+\tau)^{-\frac{5}{4}}\Lambda(\tau)d\tau}\|(n_0,u_0,E_0)\|_{H^4}\le C\delta. \tag{3.90}$$

This together with (3.89) also leads to

$$\int_0^t\|(n,Du)(\tau)\|_{H^4}^2d\tau\le C\delta, \tag{3.91}$$

and finally

$$\|(n,m,E)(t)\|_{H^4}\le C\delta. \tag{3.92}$$

Step 3: Closure of the estimates (3.61). The combination of (3.62), (3.65)–(3.70) and (3.92) leads to

$$\Lambda_1(t)\le C\delta+C\delta\Lambda_1(t)+C(\Lambda_1(t))^2,\quad t\in[0,T], \tag{3.93}$$

which together with the smallness of $\delta>0$ leads to the first estimate in (3.61). The proof is completed. □

The proof of Theorem 1.1 and Theorem 1.2: The global existence of smooth solution of the IVP problem for the compressible Navier-Stokes-Poisson system (1.1)–(1.4) follows from the short time existence theory, the uniformly a-priori estimates, and the continuity argument. The time-decay rate in Theorem 1.1 follows from the Lemma 3.3. The optimal time-decay rate in Theorem 1.2 follows from the combination of the Proposition 3.2, and the uniform estimates (3.61), (3.62), (3.67), and (3.70).

4 L^p- time decay rate

4.1 L^p decay rate for linear semigroup

In this section, we investigate the L^p- time decay rate for the solution of linearized NSP system, with $p \in [2, \infty]$. To this end, we need to analyze the Green's function $G = e^{tB}$ that formed by the linearized NSP system.

We have the L^p time decay rate for the linear semigroup as follows.

Lemma 4.1 (L^p decay rate) *Let $(n, m) = G * U_0$ and $E = L * U_0$ with N, M, L defined by (3.38) and (3.43), then we have*

$$\|D_x^\alpha n(t)\|_{L^p} = \|D_x^\alpha (N * U_0)(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}(\|U_0\|_{L^1} + \|D_x^\alpha U_0\|_{L^p}),$$
$$\|D_x^\alpha m(t)\|_{L^p} = \|D_x^\alpha (M * U_0)(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}(\|U_0\|_{L^1} + \|D_x^\alpha U_0\|_{L^p}),$$
$$\|D_x^\alpha E(t)\|_{L^p} = \|D_x^\alpha (L * U_0)(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}(\|U_0\|_{L^1} + \|D_x^\alpha U_0\|_{L^p}),$$

where $p \in [2, \infty]$, and the constant $C > 0$ independent of time.

To prove Lemma 4.1, we shall make use of following two lemmas [8].

Lemma 4.2 (L^p multiplier,[8]) *Let $n \ge 1$ and assume that $\widehat{f}(\xi) \in L^\infty \cap C^{n+1}(R^n/\{0\})$, with*

$$|D_\xi^\alpha \widehat{f}(\xi)| \le C' \begin{cases} |\xi|^{-|\alpha|+\sigma_1} & |\xi| \le R;\ |\alpha| = n, \\ |\xi|^{-|\alpha|-\sigma_2} & |\xi| \ge R;\ |\alpha| = n-1,\ n,\ n+1, \end{cases} \tag{4.1}$$

where $\sigma_1, \sigma_2 > 0$ and $n > 2 - 2\sigma_2$. Then $\widehat{f}(\xi)$ is continuous at both 0 and ∞, and

$$f = m_1 + m_2 \delta,$$

where $m_1 \in L^1(R^n)$ satisfies $\|m_1\| \le C(C')$, m_2 is the constant

$$m_2 = (2\pi)^{-\frac{n}{2}} \lim_{|\xi| \to \infty} \widehat{f}(\xi),$$

and δ is the Dirac distribution. In particular, $\widehat{f}(\xi)$ is a strong L^p multiplier, $1 \le p \le \infty$, in sense that, for any $g \in L^p$,

$$\|f * g\|_{L^p} \le C\|g\|_{L^p}, \quad 1 \le p \le \infty,$$

where $C > 0$ depends on $|m_2| \le \|\widehat{f}\|_{L^\infty}$ and the constant C'.

Lemma 4.3 ([8]) *Let $\widehat{g}(\xi, t) = \widehat{K}(\xi, t)\widehat{f}(\xi)$ with $\widehat{K}(\xi, t) = e^{-c_5|\xi|^2 t}$ and $\widehat{f}(\xi) \in L^\infty \cap C^{n+1}(\mathbb{R}^n)$ satisfying*

$$|D_\xi^\beta \widehat{f}(\xi)| \le C'|\xi|^{-|\beta|}, \quad |\beta| \le n+1. \tag{4.2}$$

Then, for any $t > 0$ $D_x^\alpha g(\cdot, t) \in L^p(\mathbb{R}^n)$ for all α and $p \in [1, \infty]$, and it holds

$$\|D_x^\alpha g(\cdot, t)\|_{L^p(\mathbb{R}^n)} \le C(|\alpha|) t^{\frac{n}{2}(\frac{1}{p}-1)-\frac{\alpha}{2}}. \tag{4.3}$$

<u>*The proof of Lemma 4.1*</u>: To make use of the above lemmas to prove Lemma 4.1, we decompose the Fourier transform $\widehat{N}, \widehat{M}, \widehat{L}$ into low frequency term and high frequency term below.

Define

$$\begin{cases} \widehat{\mathcal{N}} := \widehat{\mathcal{N}_1} + \widehat{\mathcal{N}_2}, \quad \widehat{\mathfrak{N}} := \widehat{\mathfrak{N}_1} + \widehat{\mathfrak{N}_2}; \quad \widehat{\mathcal{M}} := \widehat{\mathcal{M}_1} + \widehat{\mathcal{M}_2}, \\ \widehat{\mathfrak{M}} := \widehat{\mathfrak{M}_1} + \widehat{\mathfrak{M}_2}; \quad \widehat{\mathcal{L}} := \widehat{\mathcal{L}_1} + \widehat{\mathcal{L}_2}, \quad \widehat{\mathfrak{L}} := \widehat{\mathfrak{L}_1} + \widehat{\mathfrak{L}_2}, \end{cases} \tag{4.4}$$

where $(\cdot)_1 = \chi(\xi)(\cdot),\ (\cdot)_2 = (1 - \chi(\xi))(\cdot)$, and $\chi(\xi)$ is the smooth cut off function that

$$\chi(\xi) = \begin{cases} 1 & |\xi| \le R, \\ 0 & |\xi| \ge R + 1. \end{cases} \tag{4.5}$$

Thus, in terms of (3.38) and (4.4), we have the following decomposition of $(n, m) = G * U_0$ in Fourier modes

$$\hat{n} = \widehat{N} \cdot \widehat{U}_0 = \widehat{N}_1 \cdot \widehat{U}_0 + \widehat{N}_2 \cdot \widehat{U}_0 = (\widehat{\mathcal{N}}_1 + \widehat{\mathfrak{N}}_1) \cdot \widehat{U}_0 + (\widehat{\mathcal{N}}_2 + \widehat{\mathfrak{N}}_2) \cdot \widehat{U}_0, \tag{4.6}$$

$$\widehat{m} = \widehat{M_1} \cdot \widehat{U}_0 + \widehat{M_2} \cdot \widehat{U}_0 = (\widehat{\mathcal{M}}_1 + \widehat{\mathfrak{M}}_1) \cdot \widehat{U}_0 + (\widehat{\mathcal{M}}_2 + \widehat{\mathfrak{M}}_2) \cdot \widehat{U}_0. \tag{4.7}$$

Let we first analyze above higher frequency terms denoted by $\widehat{(\cdot)}_2$. Recall that

$$\lambda_0 = -\mu|\xi|^2, \tag{4.8}$$

$$\begin{aligned} \lambda_+ &= -(\mu + \tfrac{1}{2}\nu)|\xi|^2 + \tfrac{1}{2}i\sqrt{4(c^2|\xi|^2 + \lambda^{-2}) - (2\mu + \nu)^2|\xi|^4} \\ &= -(2\mu + \nu)|\xi|^2 + \frac{c^2}{2\mu + \nu} + \mathcal{O}(|\xi|^{-1}), \quad |\xi| \gg 1, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \lambda_- &= -(\mu + \tfrac{1}{2}\nu)|\xi|^2 - \tfrac{1}{2}i\sqrt{4(c^2|\xi|^2 + \lambda^{-2}) - (2\mu + \nu)^2|\xi|^4} \\ &= -\tfrac{c^2}{2\mu+\nu} + \mathcal{O}(|\xi|^{-1}), \quad |\xi| \gg 1. \end{aligned} \tag{4.10}$$

We shall prove that the higher frequency terms are L^p Fourier multipliers with an exponential time decay coefficient $Ce^{-c_5 t}$. For simplicity, we only show that $\widehat{\mathcal{N}_2}$ is an L^p Fourier multiplier at higher frequency as follows. It holds

$$\tfrac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = \tfrac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_- t} + (\lambda_- e^{\lambda_- t} - \lambda_- e^{\lambda_+ t})}{\lambda_+ - \lambda_-} = e^{\lambda_- t} + \tfrac{\lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} - \tfrac{\lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-},$$

and

$$\begin{aligned} \lambda_-(\xi) &= -(\mu + \frac{\nu}{2})|\xi|^2 - \frac{i}{2}\sqrt{4(c^2|\xi|^2 + \lambda^{-2}) - (2\mu + \nu)^2|\xi|^4} \\ &= -\tfrac{c^2}{2\mu+\nu} + \{\tfrac{c^2}{2\mu+\nu} - (\mu + \tfrac{\nu}{2})|\xi|^2\} + \tfrac{1}{2}\sqrt{(2\mu + \nu)^2|\xi|^4 - 4(c^2|\xi|^2 + \lambda^{-2})} \\ &= -\tfrac{c^2}{2\mu+\nu} + \frac{\lambda^{-2} + (\frac{c^2}{2\mu+\nu})^2}{\frac{1}{2}\sqrt{(2\mu+\nu)^2|\xi|^4 - 4(c^2|\xi|^2 + \lambda^{-2})} + (\mu + \frac{\nu}{2})|\xi|^2 - \frac{c^2}{2\mu+\nu}}, \end{aligned}$$

where

$$\sqrt{(2\mu + \nu)^2|\xi|^4 - 4(c^2|\xi|^2 + \lambda^{-2})} \sim \mathcal{O}(|\xi|^2), \qquad |\xi| \gg 1. \tag{4.11}$$

By a direct computation, it is easy to verify

$$|D_\xi^\alpha \lambda_-| \le C|\xi|^{2+\alpha}, \qquad |\xi| \gg 1, \tag{4.12}$$

which together with (4.9)–(4.10) gives rise to

$$|D_\xi^\alpha[(1-\chi(\cdot))e^{\lambda_- t}]| \le C\begin{cases} 0 & |\xi| \le R, \\ e^{-c_6 t}|\xi|^{-|\alpha|}|\xi|^{-2} & |\xi| \ge R, \end{cases} \tag{4.13}$$

where and below $R > 0$ is a given constant, and

$$|D_\xi^\alpha[(1-\chi(\cdot))\tfrac{\lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}]| \le C\begin{cases} 0 & |\xi| \le R, \\ e^{-c_6 t}|\xi|^{-|\alpha|}|\xi|^{-2} & |\xi| \ge R. \end{cases} \tag{4.14}$$

Thus, from Lemma 4.2 it follows that the inverse Fourier transform of the term $(1-\chi(\cdot))(e^{\lambda_- t} + \frac{\lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-})$ is an L^p multiplier with the coefficient $Ce^{-c_6 t}$. The other part of $\widehat{\mathcal{N}_2}$ at higher frequency can be written as

$$(1-\chi(\cdot))\tfrac{\lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = e^{-(\mu+\frac{\nu}{2})|\xi|^2 t} \cdot (\tfrac{e^{(\lambda_+ - (\mu+\frac{\nu}{2})|\xi|^2)t}}{\lambda_+ - \lambda_-}(1-\chi(\cdot))),$$

whose first term in right hand side can be looked on as the function $K(\xi, t)$ of Lemma 4.3, and the rest term satisfies the condition (4.2) due to the facts $(\lambda_+ - (\mu+\frac{\nu}{2})|\xi|^2)t \sim -c_6|\xi|^2 t$ for $|\xi| \ge 1$ and $D^k(e^{-|\xi|^2 t}) \le C|\xi|^{-k}((|\xi|^2 t) + (|\xi|^2 t)^2 + \cdots + (|\xi|^2 t)^k)e^{-|\xi|^2 t} \le C|\xi|^{-k}$. Thus, the inverse Fourier transform of above term is also an L^p multiplier with the coefficient $Ce^{-c_6 R^2 t}$. These facts imply that $\mathcal{N}_2$ is an L^p multiplier with the coefficient $Ce^{-c_7 t}$.

Applying the similar analysis to the terms $\widehat{\mathfrak{N}_2}$, $\widehat{\mathcal{M}_2}$, $\widehat{\mathfrak{M}_2}$, $\widehat{\mathcal{L}_2}$, and $\widehat{\mathfrak{L}_2}$, we can show that their inverse Fourier transform are all L^p multipliers with the constant $Ce^{-c_8 t}$. Thus taking $c_5 = \min\{c_7, c_8\}$ and then

$$\|D_x^\alpha(N_2 * f), D_x^\alpha(M_2 * f), D_x^\alpha(L_2 * f)\|_{L^p} \le Ce^{-c_5 t}\|D^\alpha f\|_{L^p}, \tag{4.15}$$

for all $|\alpha| \ge 0$, and $p \in [2, \infty]$.

To prove Lemma 4.1, we also need to deal with the corresponding lower frequency terms denoted by $\widehat{(\cdot)}_1$. Indeed, we can have

$$\|D_x^\alpha \mathcal{N}_1(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}, \qquad \|D_x^\alpha \mathfrak{N}_1(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}}, \tag{4.16}$$

$$\|D_x^\alpha \mathfrak{M}_1(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}, \qquad \|D_x^\alpha \mathcal{M}_1(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}, \tag{4.17}$$

$$\|D_x^\alpha \mathcal{L}_1(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}, \qquad \|D_x^\alpha \mathfrak{L}_1(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}, \tag{4.18}$$

for $t \ge 0$ and $p \in [2, \infty]$.

For simplicity, we only prove (4.16) and (4.17), since the analysis of (4.18) can be carried out in a similar way. By (2.17)–(2.19), we have the asymptotical approximation

$$\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}\sim \mathcal{O}(1)e^{-(\mu+\frac{1}{2}\nu)|\xi|^2t},\quad \frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}\sim \mathcal{O}(1)e^{-(\mu+\frac{1}{2}\nu)|\xi|^2t},\quad |\xi|\ll 1,$$
$$\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}\sim \mathcal{O}(1)e^{-(\mu+\frac{1}{2}\nu)|\xi|^2t},\quad |\xi|\ll 1,$$

which imply for $|\xi|\ll 1$ that

$$\begin{cases} |\widehat{\mathcal{N}_1}|\sim \mathcal{O}(e^{-c_9|\xi|^2t}),\quad |\widehat{\mathfrak{N}_1}|\sim \mathcal{O}(e^{-c_9|\xi|^2t}|\xi|),\\ |\widehat{\mathfrak{M}_1}|\sim \mathcal{O}(e^{-c_9|\xi|^2t}\dfrac{1}{|\xi|}),\quad |\widehat{\mathcal{M}_1}|\sim \mathcal{O}(1)e^{-c_9|\xi|^2t}. \end{cases} \tag{4.19}$$

Thus, by Hausdroff-Young's inequality, we can have

$$\|D_x^\alpha\mathcal{N}_1(t)\|_{L^p}\le C\{\int_{|\xi|\le\eta}||\xi|^{|\alpha|}e^{-c_9|\xi|^2t}|^qd\xi\}^{\frac{1}{q}}\le Ct^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}, \tag{4.20}$$
$$\|D_x^\alpha\mathfrak{N}_1(t)\|_{L^p}\le C\{\int_{|\xi|\le\eta}||\xi|^{|\alpha|+1}e^{-c_9|\xi|^2t}|^qd\xi\}^{\frac{1}{q}}\le Ct^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}}, \tag{4.21}$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $p\in[2,\infty]$, and

$$\|D_x^\alpha\mathfrak{M}_1(t)\|_{L^p}\le C\{\int_{|\xi|\le\eta}||\xi|^{|\alpha|-1}e^{-c_9|\xi|^2t}|^qd\xi\}^{\frac{1}{q}}\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}, \tag{4.22}$$
$$\|D_x^\alpha\mathcal{M}_1(t)\|_{L^p}\le C\{\int_{|\xi|\le\eta}||\xi|^{|\alpha|}e^{-c_9|\xi|^2t}|^qd\xi\}^{\frac{1}{q}}\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}. \tag{4.23}$$

These give rise to (4.16) and (4.17).

Combining the estimates (4.16)–(4.18) and (4.15), we finally have for $t>0$ that

$$\begin{aligned}\|D_x^\alpha(N*f)(t)\|_{L^p}=&\|D_x^\alpha((N_1+N_2)*f)\|_{L^p}\\ \le&C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}\|f\|_{L^1}+Ce^{-c_6t}\|D_x^\alpha f\|_{L^p}\\ \le&C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}(\|f\|_{L^1}+\|D_x^\alpha f\|_{L^p}),\qquad p\in[2,\infty],\end{aligned}$$

where $N_i=\mathcal{N}_i+\mathfrak{N}_i$, $i=1,2$, and similarly

$$\|D_x^\alpha(M*f)(t)\|_{L^p}\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}(\|f\|_{L^1}+\|D_x^\alpha f\|_{L^p}),\quad p\in[2,\infty],$$
$$\|D_x^\alpha(L*f)(t)\|_{L^p}\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}(\|f\|_{L^1}+\|D_x^\alpha f\|_{L^p}),\quad p\in[2,\infty].$$

The proof of Lemma 4.1 is completed. □

In addition to Lemma 4.1, we can also have following estimates, which can be shown in a similar procedure as in the proof of Lemma 4.1 and are applicable to the L^p time decay rate of solution of the original IVP problem for the nonlinear Navier-Stokes-Poisson system. Indeed, we have

Lemma 4.4 *It holds for the Green's functions $\mathfrak{N}$, $\mathcal{M}$, and $\mathfrak{L}$ defined by (3.39)–(3.41) that*

$$\|D_x^\alpha(\mathfrak{N} * f)(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}}(\|f\|_{L^1} + \|D_x^\alpha f\|_{L^p}), \tag{4.24}$$

$$\|D_x^\alpha(\mathfrak{N} * f)(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}}(\|f\|_{L^2} + \|D_x^\alpha f\|_{L^p}), \tag{4.25}$$

$$\|D_x^\alpha(\mathcal{M} * f)(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}(\|f\|_{L^1} + \|D_x^\alpha f\|_{L^p}), \tag{4.26}$$

$$\|D_x^\alpha(\mathcal{M} * f)(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})-\frac{|\alpha|}{2}}(\|f\|_{L^2} + \|D_x^\alpha f\|_{L^p}), \tag{4.27}$$

$$\|D_x^\alpha(\mathfrak{L} * f)(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}(\|f\|_{L^1} + \|D_x^\alpha f\|_{L^p}), \tag{4.28}$$

$$\|D_x^\alpha(\mathfrak{L} * f)(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})-\frac{|\alpha|}{2}}(\|f\|_{L^2} + \|D_x^\alpha f\|_{L^p}), \tag{4.29}$$

with $p \in [2,\infty]$, $|\alpha| = k \geq 0$, and the constant $C > 0$ independent of time, so long as $f \in L^2(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ or $f \in L^1(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$.

Proof: The estimates (4.24), (4.26) and (4.28) follow directly from those of (4.16)–(4.18) and the related estimates for higher frequency terms. We need only to prove (4.25), (4.27) and (4.29). Indeed, it holds for $p \in (2,\infty]$ that

$$\|D_x^\alpha(\mathfrak{N} * f)(t)\|_{L^p} \leq \|D_x^\alpha(\mathfrak{N}_1 * f)\|_{L^p} + \|D_x^\alpha(\mathfrak{N}_2 * f)\|_{L^p} \leq \|D_x^\alpha(\mathfrak{N}_1 * f)\|_{L^p} + Ce^{-c_7 t}\|D_x^\alpha f\|_{L^p}$$

and

$$\begin{aligned}
\|D_x^\alpha(\mathfrak{N}_1 * f)(t)\|_{L^p} \leq& C\|(i\xi)^\alpha \widehat{\mathfrak{N}_1}\hat{f}\|_{L^q} \\
\leq& C\Big[\Big(\int_{|\xi|<\eta}(|\xi|^{|\alpha|+1}e^{-c_5|\xi|^2 t})^{q\cdot\frac{2}{2-q}}d\xi\Big)^{\frac{2-q}{2}} \cdot (\||\widehat{f}|^q\|_{L^{\frac{2}{q}}})\Big]^{\frac{1}{q}} \\
\leq& C\|f\|_{L^2}(1+t)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}}
\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$. As for $p=2$, a similar direct computation gives rise to

$$\|D_x^\alpha(\mathfrak{N}_1 * f)(t)\|_{L^2} \leq C\|(i\xi)^\alpha \widehat{\mathfrak{N}_1}\hat{f}\|_{L^2} \leq C(1+t)^{-\frac{1}{2}-\frac{|\alpha|}{2}} \cdot \|f\|_{L^2}. \tag{4.30}$$

Thus, we obtain the estimate (4.25). The proof of (4.27) and (4.29) can be completed in a similar fashion and the details are omitted. □

4.2 L^p decay rate for nonlinear system

In this subsection, we show the L^p ($p \in [2,\infty]$) time decay estimates for the original nonlinear problem. We can verify that the expression of the solution of the IVP problem (3.27)–(3.30) is

$$n = N * U_0 + \int_0^t \mathfrak{N}(t-\tau) * Q(U)(\tau)d\tau, \tag{4.31}$$

$$m = M * U_0 + \int_0^t \mathcal{M}(t-\tau) * Q(U)(\tau)d\tau, \tag{4.32}$$

$$E = L * U_0 + \int_0^t \mathfrak{L}(t-\tau) * Q(U)(\tau) d\tau. \tag{4.33}$$

We have Theorem 1.3 concerned with the L^p time decay of strong solutions below.

Lemma 4.5 *Under the assumptions of Theorem 1.3, the global solutions (n, m, E) with $E = \nabla\Phi$ of the IVP problem (3.27)–(3.30) satisfies*

$$\|n(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}(\|U_0\|_{L^1} + \|U_0\|_{L^p}), \tag{4.34}$$

$$\|(m,E)(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}}(\|U_0\|_{L^1} + \|U_0\|_{L^p}), \tag{4.35}$$

with $p \in [2, \infty]$, $U_0 = (n_0, m_0)$, and the constant $C > 0$ independent of time.

Proof: By Lemma 3.3, we have for $l \ge 5$ that

$$\begin{aligned} &\|D^k n\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\delta, \quad k = 0,1,2, && \|D^3 n\|_{L^2} \le C(1+t)^{-1}\delta, \\ &\|D^k (m,E)\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\delta, \quad k = 0,1,2, && \|D^3 n\|_{L^2} \le C(1+t)^{-1}\delta, \end{aligned}$$

where $\delta =: \|(\rho_0 - \bar{\rho}, m_0)\|_{H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)} > 0$ as defined in Theorem 1.3.

Noticing that the nonlinear term $Q(U) = \nabla \cdot H$ contains mainly those like $\mathcal{O}\{mD^2 n, Dm \cdot Dn, nD^2 m, mDm, nDn, nE\}$, and using Sobolev embedding theorem, we can obtain

$$\|Q(U)\|_{L^1} \le C\delta^2(1+t)^{-1}, \|Q(U)\|_{L^p} \le C\delta^2(1+t)^{-\frac{7}{4}}, \quad p \in [2,\infty]. \tag{4.36}$$

Thus, by (4.31), (4.36), Lemma 4.1 and Lemma 4.4, we are able to show

$$\begin{aligned} \|n(t)\|_{L^p} \le &\|N * U_0(t)\|_{L^p} + \int_0^t \|\mathfrak{N}(t-\tau) * Q(U)(\tau)\|_{L^p} d\tau \\ \le &C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}(\|U_0\|_{L^1} + \|U_0\|_{L^p}) + C\int_0^t \|\mathcal{N}(t-\tau) * Q(U)(\tau)\|_{L^p} d\tau \\ \le &C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}(\|U_0\|_{L^1} + \|U_0\|_{L^p}) \\ &+ C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}}(\|Q(U)\|_{L^1} + \|Q(U)\|_{L^p}) d\tau \\ &+ C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}(\|Q(U)\|_{L^2} + \|Q(U)\|_{L^p}) d\tau \\ \le &C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}(\|U_0\|_{L^1} + \|U_0\|_{L^p}) + C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\varepsilon}\delta^2 \\ \le &C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}\delta, \end{aligned} \tag{4.37}$$

and

$$\|m(t)\|_{L^p} \le \|M * U_0(t)\|_{L^p} + \int_0^t \|\mathcal{M}(t-\tau) * Q(U)(\tau)\|_{L^p} d\tau$$

$$
\begin{aligned}
&\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}}(\|U_0\|_{L^1}+\|U_0\|_{L^p})+C\int_0^t\|\mathcal{M}(t-\tau)*Q(U)(\tau)\|_{L^p}d\tau\\
&\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}}(\|U_0\|_{L^1}+\|U_0\|_{L^p})\\
&\quad+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{2}(1-\frac{1}{p})}(\|Q(U)\|_{L^1}+\|Q(U)\|_{L^p})d\tau\\
&\quad+C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})}(\|Q(U)\|_{L^2}+\|Q(U)\|_{L^p})d\tau\\
&\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}}(\|U_0\|_{L^1}+\|U_0\|_{L^p})+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{2}(1-\frac{1}{p})}(1+\tau)^{-1}d\tau\\
&\quad+C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})}(1+\tau)^{-\frac{7}{4}}d\tau\\
&\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}}\delta+C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\varepsilon}\delta^2\\
&\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}}\delta, \qquad (4.38)
\end{aligned}
$$

where $p\in[2,\infty]$, $C>0$ is a constant independent of time, and $\varepsilon>$ is a small but fixed constant.

The estimates of $\|E\|_{L^p}$ can be obtained in terms of $\|m\|$ and Riesz potential as

$$
\|E(t)\|_{L^p}\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}}(\|U_0\|_{L^1}+\|U_0\|_{L^p}),\quad p\in[2,\infty]. \qquad (4.39)
$$

The proof of the Lemma 4.5 is completed. □

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